SANDIA REPORT

Trilinos Tutorial

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Abstract

The Trilinos Project is an effort to facilitate the design, development, integration and ongoing support of mathematical software libraries. The goal of the Trilinos Project is to develop parallel solver algorithms and libraries within an object-oriented software framework for the solution of large-scale, complex multiphysics engineering and scientific applications. The emphasis is on developing robust, scalable algorithms in a software framework, using abstract interfaces for flexible interoperability of components while providing a full-featured set of concrete classes that implement all the abstract interfaces.

This document introduces the use of Trilinos, version 4.0. The presented material includes, among others, the definition of distributed matrices and vectors with Epetra, the iterative solution of linear systems with AztecOO, incomplete factorizations with IF-PACK, multilevel and domain decomposition preconditioners with ML, direct solution of linear system with Amesos, eigenvalues and eigenvectors computations with Anasazi, and iterative solution of nonlinear systems with NOX.

The tutorial is a self-contained introduction, intented to help computational scientists effectively apply the appropriate Trilinos package to their applications. Basic examples are presented that are fit to be imitated.

This document is a companion to the Trilinos User's Guide [20] and Trilinos Development Guides [21, 22]. Please note that the documentation included in each of the Trilinos' packages is of fundamental importance.

Acknowledgments

The authors would like to acknowledge the support of the ASCI and LDRD programs that funded development of Trilinos.

The developers of each Trilinos' package are acknowledged for providing excellent documentation and examples, without which this document would have never been possible. In particular, we thank Heidi Thornquist for the contributions to the Teuchos chapter, and Erik Boman for the chapter on partitioning and load balancing.

Trilinos Tutorial

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1 Introduction

1.1 Getting Started

The Trilinos framework uses a two level software structure that connects a system of *packages*. A Trilinos package is an integral unit, usually developed to solve a specific task, by a (relatively) small group of experts. Packages exist beneath the Trilinos top level, which provides a common look-and-feel. Each package has its own structure, documentation and set of examples, and it is possibly available independently of Trilinos. However, each package is even more valuable when combined with other Trilinos packages.

Trilinos is a large software project, and currently about twenty packages are included. The entire set of packages covers a wide range of numerical methods for large scale computing, as well as a large set of utilities to improve the development of software for scientific computing.

Clearly, a full understanding all the functionalities of the Trilinos packages requires time. Each package offers sophisticated features, difficult to "unleash" at a first sight. Besides that, a detailed description of each Trilinos package is beyond the scope of this document. For these reasons, the goal of this tutorial is to ensure that users have the background to make good use of the extensive documentation contained in each package.

We will describe the following subset of the Trilinos packages.

- **Epetra**. The package defines the basic classes for distributed matrices and vectors, linear operators and linear problems. Epetra classes are the common language spoken by all the Trilinos packages (even if some packages can "speak" other languages). Each Trilinos package accepts as input Epetra objects. This allows powerful combinations among the various Trilinos functionalities.

- **Triutils**. This is a collection of utilities that are useful in software development. Here, we present a command line parser and a matrix generator, that are used throughout this document to define example matrices.

- **AztecOO**. This is a linear solver package based on preconditioned Krylov methods. Aztec users will find that AztecOO supports all the Aztec interfaces and functionality, and also provides significant new functionality.

- **IFPACK**. The package performs various incomplete factorizations, and is here used with AztecOO.

- **Teuchos**. This is a collection of classes that can be essential for advanced code development.

- **ML**. The algebraic multilevel and domain decomposition preconditioner package provides scalable preconditioning capabilities for a variety of problems. It is here used as a preconditioner for AztecOO solvers.

- **Amesos**. The package provides a common interface to certain sparse direct linear solvers (generally available outside the Trilinos framework), both sequential and parallel.

- **Anasazi**. The package provides a common interface to parallel eigenvalue and eigenvector solvers, for both symmetric and non-symmetric linear problems.

- **NOX**. This is a collection of nonlinear solvers, designed to be easily integrated into an application and used with many different linear solvers.

- **Didasko**. This package contains all the examples reported in this tutorial. The sources of the examples can be found in the subdirectory
  `<your-trilinos-home>/packages/didasko/examples`.

Table 1 gives a partial overview of what can be accomplished using Trilinos.

Remark 1. *As already pointed out, Epetra objects are meant to be the "common language" spoken by all the Trilinos packages, and are a natural starting point. For new users, Chapters 2-4 are a prerequisite to the later chapters. Chapters 5 is not essential to understand Trilinos, but the functionalities there presented are used in this document as a starting point for many examples. One of the classes described in Chapter 8, the Teuchos::ParameterList, is later used in Chapters 9 and 10. Chapter 6 should be read before Chapters 7 and 9 (even if both IFPACK and ML can be compiled and run without AztecOO).*

The only prerequisites assumed in this tutorial are some familiarities with numerical methods for PDEs, and with iterative linear and nonlinear solvers. Although not strictly necessary, the reader is assumed to have some familiarity with distributed memory computing and, to a lesser extent, with MPI[1].

Note that this tutorial is not a substitute for individual packages' documentation. Also, for an overview of all the Trilinos packages, the Trilinos philosophy, and a description of the packages provided by Trilinos, the reader is referred to [17]. Developers should also consider the Trilinos Developers' Guide, which addresses many topics, including the development tools used by Trilinos' developers, and a description of how to include a new package[2].

1.2 Installation

To obtain Trilinos, please follow the instructions at the web site

---

[1]Although almost no explicit MPI instructions are required in a Trilinos code, the reader should be aware of the basic concepts of message passing, like the definition of a communicator.

[2]Trilinos provides a variety of services to a developer wanting to integrate a package into Trilinos. The services include Autoconf [11], Automake [12] and Libtool [13]. The tools provide a robust, full-featured set of tools for building software across a broad set of platforms. The tools are not officially standards, but are widely used. All existing Trilinos packages use Autoconf and Automake. Libtool support will be added in future releases.

Service provided/Task performed	Package	Tutorial
Advanced serial dense or sparse matrices:	Epetra	Chapter 3
Advanced utilities for Epetra vectors and sparse matrices:	EpetraExt	–
Templated distributed vectors and sparse matrices:	Tpetra*	–
Distributed sparse matrices:	Epetra	–
Solve a linear system with preconditioned Krylov accelerators, CG, GMRES, Bi-CGSTAB, TFQMR:	AztecOO, Belos*	Chapter 6
Incomplete Factorizations:	AztecOO, IFPACK	Chapter 7
Multilevel preconditioners:	ML	Chapter 9
"Black-box" smoothed aggregation preconditioners:	ML	Section 9.4
One-level Schwarz preconditioner (overlapping domain decomposition):	AztecOO, IFPACK	Chapter 7
Two-level Schwarz preconditioner, with coarse matrix based on aggregation:	AztecOO+ML	Section 9.5
Systems of nonlinear equations:	NOX	Chapter 12
Interface with various direct solvers, as UMF-PACK, MUMPS, SuperLU_DIST and ScaLA-PACK :	Amesos	Chapter 10
Eigenvalue problems for sparse matrices:	Anasazi	Chapter 11
Complex linear systems (using equivalent real formulation):	Komplex*	–
Segregated and block preconditioners (e.g., incompressible Navier-Stokes equations):	Meros*	–
Light-weight interface to BLAS and LAPACK:	Epetra	Chapter 3
Templated interface to BLAS and LAPACK, arbitrary-precision arithmetic, parameters' list, smart pointers:	Teuchos	Section 8.4
Definition of abstract interfaces to vectors, linear operators, and solvers:	TSF*, TSFCore*, TSFExtended*	–
Generation of test matrices	Triutils	Section 5.2

Table 1. Partial overview of intended uses of Trilinos. *: not covered in this tutorial.

```
http://software.sandia.gov/Trilinos
```

Trilinos has been compiled on a variety of architectures, including various flavors of Linux, Sun Solaris, SGI Irix, DEC, Mac OS X, ASCI Red, and many others. Trilinos has been designed to support parallel applications. However, it also compiles and runs on serial computers. Detailed comments on the installation, and an exhaustive list of FAQs, may be found at the web pages:

```
http://software.sandia.gov/Trilinos/installing_manual.html
http://software.sandia.gov/Trilinos/faq.html
```

After obtaining Trilinos, the next step is its compilation. The description here is for LINUX platforms with MPI, the compilation on other platforms being pretty similar.

Each Trilinos' package can be enabled or disabled at configuration time. A procedure one may follow in order to compile Trilinos with AztecOO, ML, IFPACK, Anasazi, NOX, Triutils, and Didasko (which contains the examples used in this tutorial) is reported below. $ indicates the shell prompt. The `tee` command is used to write the output to standard output and to the specified file, and may be omitted. More details about the installation of Trilinos can be found in [20].

```
$ cd <your-trilinos-directory>
$ mkdir LINUX_MPI
$ cd LINUX_MPI
$ ../configure --prefix=<your-trilinos-directory>/LINUX_MPI \
  --enable-mpi --with-mpi-compilers \
  --disable-default-packages        \
  --enable-teuchos    \
  --enable-epetra     \
  --enable-aztecoo    \
  --enable-triutils   \
  --enable-amesos     \
  --enable-ifpack     \
  --enable-ml         \
  --enable-nox        \
  --enable-nox-epetra \
  --enable-didasko    | tee configure_LINUX_MPI.log
$ make | tee make_LINUX_MPI.log
$ make install | tee make_install_LINUX_MPI.log
```

For more details, simply type

```
$ <your-trilinos-directory>/configure --help
```

For serial configuration, simply delete the `--enable-mpi --with-mpi-compilers` options (and possibly change the name of the installation directory from `LINUX_MPI` to something like `LINUX_SERIAL`).

Remark 2. *All Trilinos packages can be build to run with or without MPI. If MPI is enabled (using* `--enable-mpi`*), the users must know the procedure for beginning MPI jobs on their computer system(s). It may be necessary to specify on the configure line the location of MPI include files and libraries.*

Remark 3. *Compiling Trilinos may require a little bit of time! For instance, on a Pentium 4, 1.7 GHz with Intel compilers, to configure Trilinos with LOCA, NOX, AztecOO, Anasazi, IFPACK, Triutils, EpetraExt, Teuchos, ML and Epetra, required 2 minutes and 11 seconds. The compilation (without the examples) took 5 minutes and 41 seconds. The reader might find convenient to start by enabling a limited number of packages, then re-configuring Trilinos as required.*

1.3 Copyright and Licensing of Trilinos

Trilinos is released under the Lesser GPL GNU Licence.

Trilinos is copyrighted by Sandia Corporation. Under the terms of Contract DE-AC04-94AL85000, there is a non-exclusive license for use of this work by or on behalf of the U.S. Government. Export of this program may require a license from the United States Government.

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Some parts of Trilinos are dependent on a third party code. Each third party code comes with its own copyright and/or licensing requirements. It is responsibility of the user to understand these requirements.

1.4   Programming Language Used in this Tutorial

Trilinos is written in C++ (for most packages), and in C. Some interfaces are provided to FORTRAN codes (mainly BLAS and LAPACK routines). Even if limited support is included for C programs (and a more limited for FORTRAN code), to unleash the full power of Trilinos we recommend C++. All the example programs contained in this tutorial are in C++; some packages (like ML) contain examples in C.

1.5   Referencing Trilinos

The Trilinos project can be referenced by using the following BiBTeX citation information:

```
@techreport{Trilinos-Overview,
title = "{An Overview of Trilinos}",
author = "Michael Heroux and Roscoe Bartlett and Vicki Howle
Robert Hoekstra and Jonathan Hu and Tamara Kolda and
Richard Lehoucq and Kevin Long and Roger Pawlowski and
Eric Phipps and Andrew Salinger and Heidi Thornquist and
Ray Tuminaro and James Willenbring and Alan Williams ",
institution = "Sandia National Laboratories",
number = "SAND2003-2927",
year = 2003}

@techreport{Trilinos-Dev-Guide,
title = "{Trilinos Developers Guide}",
author = "Michael A. Heroux and James M. Willenbring and Robert Heaphy",
institution = "Sandia National Laboratories",
number = "SAND2003-1898",
year = 2003}

@techreport{Trilinos-Dev-Guide-II,
title = "{Trilinos Developers Guide Part II: ASCI Software Quality
Engineering Practices Version 1.0}",
author = "Michael A. Heroux and James M. Willenbring and Robert Heaphy",
institution = "Sandia National Laboratories",
number = "SAND2003-1899",
year = 2003}

@techreport{Trilinos-Users-Guide,
title = "{Trilinos Users Guide}",
author = "Michael A. Heroux and James M. Willenbring",
institution = "Sandia National Laboratories",
```

```
number = "SAND2003-2952",
year = 2003}

@techreport{Trilinos-Tutorial-5.0,
title = "{Trilinos Tutorial}",
author = "Marzio Sala and Michael A. Heroux and David D. Day",
institution = "Sandia National Laboratories",
number = "SAND2004-2189",
year = 2004}
```

The BiBTeX information is available at the web page

```
http://software.sandia.gov/Trilinos/citing.html
```

1.6   A Note on the Directory Structure

Each Trilinos package in contained in the subdirectory

```
<your-trilinos-directory>/packages
```

Each package contains sources, examples, tests and documentation subdirectories:

```
<your-trilinos-directory>/packages/<package-name>/src
<your-trilinos-directory>/packages/<package-name>/examples
<your-trilinos-directory>/packages/<package-name>/test
<your-trilinos-directory>/packages/<package-name>/doc
```

Developers' documentation is written using Doxygen[3]. For instance, to create the documentation for Epetra are

```
$ cd <your-trilinos-home>/packages/epetra/doc
$ doxygen
```

Generally, both HTML and LaTeX documentation are created by Doxygen. The browser of choice can be used to walk through the HTML documentation. To compile the LaTeX sources, the commands are:

```
$ cd <your-trilinos-home>/packages/epetra/doc/latex
$ make
```

---

[3]Copyright ©1997-2003 by Dimitri van Heesch. More information can by found at the web address `http://www.stack.nl/ dimitri/doxygen/`.

1.7 List of Trilinos Developers

The Trilinos' developers as of May 2004 are (in alphabetical order):

Roscoe A. Bartlett, Jason A. Cross, David M. Day, Robert Heaphy, Michael A. Heroux (project leader), Russell Hooper, Vicki E. Howle, Robert J. Hoekstra, Jonathan J. Hu, Tamara G. Kolda, Richard B. Lehoucq, Paul Lin, Kevin R. Long, Roger P. Pawlowski, Michael N. Phenow, Eric T. Phipps, Andrew J. Rothfuss, Marzio Sala, Andrew G. Salinger, Paul M. Sexton, Kendall S. Stanley, Heidi K. Thornquist, Ray S. Tuminaro, James M. Willenbring, and Alan Williams.

2   Working with Epetra Vectors

A vector is a fundamental data structure required by almost all numerical methods. Within the Trilinos framework, vectors are usually constructed starting from Epetra classes.

An Epetra vector may store either double-precision values (like the solution of a PDE problem, the right-hand side of a linear system, or nodal coordinates), or integer data values (such as a set of indexes or global IDs).

An Epetra vector may be either *serial* or *distributed*. Serial vectors are usually small, so that it is not convenient to distribute them across the processes. Possibly, serial vectors are replicated across the processes. On the other hand, distributed vectors tend to be significantly larger, and therefore their elements are distributed across the processors. In this latter case, users must specify the partition they intend to use. In Epetra, this is done by specifying a communicator (introduced in Section 2.1) and an Epetra object called map (introduced in Section 2.2). A map is basically a partitioning of a list of global IDs.

During the Chapter, the user will be introduced to:

- The fundamental Epetra communicator object, Epetra_Comm (in Section 2.1);

- The Epetra_Map object (in Section 2.2);

- The creation and assembly of Epetra vectors (in Sections 2.3 and 2.4). The sections also present common vector operations, such as dot products, fill with constant or random values, vector scalings and norms;

- A tool to redistributing vectors across processes (in Section 2.5).

2.1   Epetra Communicator Objects

The Epetra_Comm virtual class is an interface that encapsulates the general information and services needed for the other Epetra classes to run on serial or parallel computer. An Epetra_Comm object is required for building all Epetra_Map objects, which in turn are required for all other Epetra classes.

Epetra_Comm has two basic concrete implementations:

- Epetra_SerialComm (for serial executions);

- Epetra_MpiComm (for MPI distributed memory executions).

For most basic applications, the user can create an Epetra_Comm object using the following code fragment:

```
#include "Epetra_ConfigDefs.h"
#ifdef HAVE_MPI
#include "mpi.h"
```

```
#include "Epetra_MpiComm.h"
#else
#include "Epetra_SerialComm.h"
#endif
// .. other include files and others ...
int main( int argv, char *argv[]) {
 // .. some declarations here ...
#ifdef HAVE_MPI
  MPI_Init(&argc, &argv);
  Epetra_MpiComm Comm(MPI_COMM_WORLD);
#else
  Epetra_SerialComm Comm;
#endif
// ... other code follows ...
```

Note that the `MPI_Init()` call and the

```
#ifdef HAVE_MPI
  MPI_Finalize();
#endif
```

call, are likely to be the *only* MPI calls users have to explicitly introduce in their code.

Most of Epetra_Comm methods are similar to MPI functions. The class provides methods such as `MyPID()`, `NumProc()`, `Barrier()`, `Broadcast()`, `SumAll()`, `GatherAll()`, `MaxAll()`, `MinAll()`, `ScanSum()`. For instance, the number of processes in the communicator, `NumProc`, and the ID of the calling process, `MyPID`, can be obtained by

```
int NumProc = Comm.NumProc();
int MyPID = Comm.MyPID();
```

The file `didasko/examples/epetra/ex1.cpp` presents the use of some of the above introduced functions. For a description of the syntax, please refer to the Epetra Class Documentation.

2.2 Defining a Map

The distribution of a set of integer labels (or elements) across the processes is here called a *map*, and its actual implementation is given by the Epetra_Map class (or, more precisely, by an Epetra_BlockMap, from which Epetra_Map is derived). Basically, the class handles the definition of the:

- global number of elements in the set (called `NumGlobalElements`);

- local number of elements (called `NumMyElements`);

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```
NumGlobaElementss()
```
The total number of elements across all processes.

```
NumMyElementss()
```
The number of elements on the calling process.

```
MinAllGID()
```
The minimum global index value across all processes.

```
MaxAllGID()
```
The maximum global index value across all processes.

```
MinMyGID()
```
The minimum global index value on the calling process.

```
MaxMyGID()
```
The maximum global index value on the calling process.

```
MinLID()
```
The minimum local index value on the calling process.

```
MaxLID()
```
The maximum local index value on the calling process.

```
LinearMap()
```
Returns true if the elements are distributed linearly across processes, i.e., process 0 gets the first n/p elements, process 1 gets the next n/p elements, etc. where n is the number of elements and p is the number of processes.

```
DistributedGlobal()
```
Returns true if the element space of the map spans more than one process. This will be true in most cases, but will be false in serial cases and for objects that are created via the derived Epetra_LocalMap class.

Table 2. Some methods of the class `Epetra_Map`

- global numbering of all local elements (an integer vector of size `NumMyElements`, called `MyGlobalElements`).

There are three ways to define an map. The easiest way is to specify the global number of elements, and let Epetra decide:

```
Epetra_Map Map(NumGlobalElements,0,Comm);
```

In this case, the constructor takes the global dimension of the vector, the base index[4], and an `Epetra_Comm` object (introduced in Section 2.1). As a result, each process will be assigned a contiguous set of elements.

A second way to build the Epetra_Comm object is to furnish the local number of elements:

```
Epetra_Map Map(-1,NumMyElements,0,Comm);
```

This will create a vector of size $\sum_{i=0}^{NumProc-1}$ `NumMyElements`. Each process will get a contiguous set of elements. These two approached are coded in file `didasko/examples/epetra/ex2.cpp`.

A third more involved way to create an Epetra_Map, is to specify on each process both the number of local elements, and the global indexing of each local element. To understand this, consider the following code. A vector of global dimension 5 is split among processes `p0` and `p1`. Process `p0` owns elements 0 an 4, and process `p1` elements 1, 2, and 3.

```
#include "Epetra_Map.h"
// ...
MyPID = Comm.MyPID();
switch( MyPID ) {
case 0:
  MyElements = 2;
  MyGlobalElements = new int[MyElements];
  MyGlobalElements[0] = 0;
  MyGlobalElements[1] = 4;
  break;
case 1:
  MyElements = 3;
  MyGlobalElements = new int[MyElements];
  MyGlobalElements[0] = 1;
  MyGlobalElements[1] = 2;
  MyGlobalElements[2] = 3;
  break;
```

---

[4]The index base is the index of the lowest order element, and is usually, `0` for C or C++ arrays, and `1` for FORTRAN arrays. Epetra can indeed accept any number as index base. However, some other Trilinos package may require a C-style index base.

```
}
```

```
Epetra_Map Map(-1,MyElements,MyGlobalElements,0,Comm);
```

The complete code is reported in `didasko/examples/epetra/ex3.cpp`.

Once created, a Map object can be queried for the global and local number of elements, using

```
int NumGlobalElements = Map.NumGlobalElements();
int NumMyElements = Map.NumMyElements();
```

and for the global ID of local elements, using

```
int * MyGlobalElements = Map.MyGlobalElements();
```

that returns a pointer to the internally stored global indexing vector, or, equivalently,

```
int MyGlobalElements[NumMyElements];
Map.MyGlobalElements(MyGlobalElements);
```

that copies in the user's provided array the global indexing.

The class Epetra_Map is derived from Epetra_BlockMap. The class keeps information that describes the distribution of objects that have block elements (for example, one or more contiguous entries of a vector). This situation is common in applications like multiple-unknown PDE problems. A variety of constructors are available for the class. An example of the use of block maps is reported in `didasko/examples/epetra/ex23.cpp`.

Note that different maps may coexist in the same part of the code. The user may define vectors with different distributions (even for vectors of the same size). Two classes are provided to transfer data from one map to an other: Epetra_Import and Epetra_Export (see Section 2.5).

Remark 4. *Most Epetra objects overload the* << *operator. For example, to visualize information about the* Map*, one can simply write*

```
cout << Map;
```

We have constructed very basic map objects. More general objects can be constructed as well. First, element numbers are only labels, and they do not have to be consecutive. This means that we can define a map with elements 1, 100 and 10000 on process 0, and elements 2, 200 and 20000 on process 1. This map, composed by 6 elements, is perfectly legal. Second, each element can be assigned to more than one process. Examples
`didasko/examples/epetra/ex20.cpp` and
`didasko/examples/epetra/ex21.cpp` can be used to better understand the potential of Epetra_Maps.

Remark 5. *The use of "distributed directory" technology facilitates arbitrary global ID support.*

2.3 Creating and Assembling Serial Vectors

Within Epetra, it is possible to define *sequential* vectors for serial and parallel platforms. A sequential vector is a vector which, in the opinion of the programmer, does not need to be partitioned among the processes. Note that each process defines its own sequential vectors, and that changing an element of this vector on this process will *not* directly affect the vectors stored on other processes (if any have been defined).

The class Epetra_SerialDenseVector enables the construction and use of real-valued, double precision dense vectors. The Epetra_SerialDenseVector class provides convenient vector notation but derives all significant functionality from Epetra_SerialDenseMatrix class (see Section 3.1). The following instruction creates a sequential double-precision vector containing `Length` elements:

```
#include "Epetra_SerialDenseVector.h"
Epetra_SerialDenseVector DoubleVector(Length);
```

Other constructors are available, as described in the Epetra Class Documentation. Integer vectors can be created as

```
#include "Epetra_IntSerialDenseVector.h"
Epetra_SerialIntDenseVector IntVector(Length);
```

We recomment Epetra_SerialDenseVector and Epetra_SerialIntDenseVector instead of more common C++ allocations (using `new`), because Epetra serial vectors automatically delete the allocated memory when destructed, avoiding possible memory leaks.

The vector can be filled using the `[]` or `()` operators. Both methods return a reference to the specified element of the vector. However, using `()`, bound checking is enforced. Using using `[]`, no bounds checking is done unless Epetra is compiled with `EPETRA_ARRAY_BOUNDS_CHECK`.

Remark 6. *To construct replicated Epetra objects on distributed memory machines, the user may consider the class Epetra_LocalMap. The class constructs the replicated local objects and keeps information that describe the distribution.*

The file `didasko/examples/epetra/ex4.cpp` illustrates basic operations on dense vectors.

2.4 Creating and Assembling a Distributed Vector

A distributed object is an entity whose elements are partitioned across more than one process. Epetra's distributed objects (derived from the Epetra_DistObject class) are created from a Map. For example, a distributed vector can be constructed starting from an Epetra_Map (or Epetra_BlockMap) with an instruction of type

```
Epetra_Vector x(Map);
```

(We shall see that this dependency on Map objects holds for all distributed Epetra objects.) This constructor allocates space for the vector and sets all the elements to zero. A copy constructor may be used as well:

```
Epetra_Vector y(x);
```

A variety of sophisticated constructors are indeed avaiable. For instance, the user can pass a pointer to an array of double precision values,

```
Epetra_Vector x(Copy,Map,LocalValues);
```

Note the word `Copy` is input to the constructor. It specifies the Epetra_CopyMode, and refers to many Epetra objects. In fact, Epetra allows two data access modes:

1. `Copy`: allocate memory and copy the user-provided data. In this mode, the user data is not needed be the new Epetra_Vector after construction;

2. `View`: create a "view" of the user's data. The user data is assumed to remain untouched for the life of the vector (or modified carefully). From a data hiding perspective, View mode is very dangerous. But is is often the only way to get the required performance. Therefore, users are strongly encouraged to develop code using the Copy mode. Only use View mode as needed in a secondary performance optimization phase. To use the View mode, the user has to define the vector entries using a (double) vector (of appropriate size), than construct an Epetra_Vector with an instruction of type

   ```
   Epetra_Vector z(View,Map,z_values);
   ```

   where `z_values` is a pointer a double array containing the values for `z`.

   To set a locally owned element of a vector, ont can use the `[ ]` operator, regardless of how a vector has been created. For example,

```
x[i] = 1.0*i;
```

where `i` is in the local index space.

Epetra also defines some functions to set vector elements in local or global index space. `ReplaceMyValues` or `SumIntoMyValues` will replace or sum values into a vector with a given indexed list of values, with indexes in the *local* index space; `ReplaceGlobalValues` or `SumIntoGlobalValues` will replace or sum values into a vector with a given indexed list of values in the *global* index space (but locally owned). It is important to note that no process may set vector entries locally owned by another process. In other words, both global and local insert and replace functions refer to the part of a vector assigned to the calling process.

Intra-process communications can be (easily) performed using Import and Export objects, covered in Section 2.5.

The user might need (for example, for reasons of computational efficiency) to work on Epetra_Vectors as if they were `double *` pointers. File
`didasko/examples/epetra/ex5.cpp`
shows the use of `ExtractCopy()`. `ExtractCopy` does not give access to the vector elements, but only copies them into the user-provided array. The user must commit those changes to the vector object, using, for instance, `ReplaceMyValues`.

A further computationally efficient way, is to extract a "view" of the (multi-)vector internal data. This can be done as follows, using method `ExtractView()`. Let z be an Epetra_Vector.

```
double * z_values;
z.ExtractView( &z_values );
for( int i=0 ; i<MyLength ; ++i ) z_values[i] *= 10;
```

In this way, modifying the values of `z_values` will affect the internal data of the Epetra_Vector z. An example of the use of `ExtractView` is reported in file
`didasko/examples/epetra/ex6.cpp`.

Remark 7. *The class Epetra_Vector is derived from Epetra_MultiVector. Roughly speaking, a multi-vector is a collection of one or more vectors, all having the same length and distribution. File* `didasko/examples/epetra/ex7.cpp` *illustrates use of multi-vectors.*

The user can also consider the function `ResetView`, which allows a (very) light-weight replacement of multi-vector values, created using the Epetra_DataMode `View`. Note that no checking is performed to see if the values passed in contain valid data. This method can be extremely useful in the situation where a vector is needed for use with an Epetra operator or matrix, and the user is not passing in a multi-vector. Use this method with caution as it could be extremely dangerous. A simple example is reported in
`didasko/examples/epetra/ex8.cpp`

It is possible to perform a certain number of operations on vector objects. Some of them are reported in Table 3. Example `didasko/examples/epetra/ex18.cpp` works with some of the functions reported in the table.

2.5 Epetra_Import and Epetra_Export classes

The Epetra_Import and Epetra_Export classes suites apply off-processor communication. Epetra_Import and Epetra_Export are used to construct a communication plan that can be called repeatedly by computational classes such the Epetra multi-vectors of the Epetra matrices.

Currently, those classes have one constructor, taking two Epetra_Map (or Epetra_BlockMap) objects. The first map specifies the global IDs that are owned by the calling processor. The second map specifies the global IDs of elements that we want to import later.

```
int NumMyELements()
```
returns the local vector length on the calling processor

```
int NumGlobalElements()
```
returns the global length

```
int Norm1(double *Result) const
```
returns the 1-norm (defined as $\sum_i^n |x_i|$ (see also `Norm2` and `NormInf`)

```
Normweigthed(double *Result) const
```
returns the 2-norm, defined as $\sqrt{\frac{1}{n}\sum_{j=1}^n (w_j x_j)^2})$

```
int Dot(const Epetra MultiVector A, double *Result) const
```
computes the dot product of each corresponding pair of vectors

```
int Scale(double ScalarA, const Epetra MultiVector &A
```
Replace multi-vector values with scaled values of A, `this=ScalarA*A`

```
int MinValue(double *Result) const
```
compute minimum value of each vector in multi-vector (see also `MaxValue` and `MeanValue`)

```
int PutScalar(double Scalar)
```
Initialize all values in a multi-vector with constant value

```
int Random()
```
set multi-vector values to random numbers

Table 3. Some methods of the class `Epetra Vector`

Using an Epetra_Import object means that the calling process knows what it wants to receive, while an Epetra_Export object means that it knows what it wants to send. An Epetra_Import object can be used to do an Export as a reserve operation (and equivalently an Epetra_Export can be used to do an Import). In the particular case of bijective maps, either Epetra_Import or Epetra_Export is appropriate.

To better illustrate the use of these two classes, we present the following example. Suppose that the double-precision distributed vector x of global length 4, is distributed over two processes. Process 0 own elements 0,1,2, while process 1 owns elements 1,2,3. This means that elements 1 and 2 are replicated over the two processes. Suppose that we want to bring all the components of x to process 0, summing up the contributions of elements 1 and 2 from the 2 processes. This is done in the following example (the complete code is reported in didasko/examples/epetra/ex9.cpp).

```
int NumGlobalElements = 4; // global dimension of the problem

int NumMyElements; // local elements
Epetra_IntSerialDenseVector MyGlobalElements;

if( Comm.MyPID() == 0 ) {
  NumMyElements = 3;
  MyGlobalElements.Size(NumMyElements);
  MyGlobalElements[0] = 0;
  MyGlobalElements[1] = 1;
  MyGlobalElements[2] = 2;
} else {
  NumMyElements = 3;
  MyGlobalElements.Size(NumMyElements);
  MyGlobalElements[0] = 1;
  MyGlobalElements[1] = 2;
  MyGlobalElements[2] = 3;
}

// create a double-precision map
Epetra_Map Map(-1,MyGlobalElements.Length(),
               MyGlobalElements.Values(),0, Comm);

// create a vector based on map
Epetra_Vector x(Map);
for( int i=0 ; i<NumMyElements ; ++i )
  x[i] = 10*( Comm.MyPID()+1 );
cout << x;
```

```
// create a target map, in which all the elements are on proc 0
int NumMyElements_target;

if( Comm.MyPID() == 0 )
  NumMyElements_target = NumGlobalElements;
else
  NumMyElements_target = 0;

Epetra_Map TargetMap(-1,NumMyElements_target,0,Comm);

Epetra_Export Exporter(Map,TargetMap);

// work on vectors
Epetra_Vector y(TargetMap);

y.Export(x,Exporter,Add);
cout << y;
```

Running this code with 2 processors, the output will be approximatively the following:

```
[msala:epetra]> mpirun -np 2 ./ex31.exe
Epetra::Vector
    MyPID              GID                 Value
        0                0                      10
        0                1                      10
        0                2                      10
Epetra::Vector
        1                1                      20
        1                2                      20
        1                3                      20
Epetra::Vector
Epetra::Vector
    MyPID              GID                 Value
        0                0                      10
        0                1                      30
        0                2                      30
        0                3                      20
```

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3 Working with Epetra Matrices

Epetra contains several matrix classes. Epetra matrices can be defined to be either *serial* or *distributed*. A serial matrix could be the matrix corresponding to a given element in a finite-element discretization, or the Hessemberg matrix in the GMRES method. Those matrices are of (relatively) small size, so that it is not convenient to distribute them across the processes.

Other matrices, e.g. the linear system matrices, must be distributed to obtain scalability. For distributed sparse matrices, the basic Epetra class is Epetra_RowMatrix, meant for double-precision matrices with row access. Epetra_RowMatrix is a pure virtual class. The classes that are derived from `Epetra_RowMatrix` include:

- `Epetra_CrsMatrix` for point matrices;

- `Epetra_VbrMatrix` for block matrices (that is, for matrices which have a block structure, for example the ones deriving from the discretization of a PDE problem with multiple unknowns for node);

- `Epetra_FECrsMatrix` and `Epetra_FEVbrMatrix` for matrices arising from FE discretizations.

The purpose of the Chapter is to review the allocation and assembling of different types of matrices as follows:

- The creation of (serial) dense matrices (in Section 3.1);

- The creation of sparse point matrices (in Section 3.2);

- The creation of sparse block matrices (in Section 3.3);

- The insertion of non-local elements using finite-element matrices (in Section 3.4).

3.1 Serial Dense Matrices

Epetra supports sequential dense matrices with the class Epetra_SerialDenseMatrix. A possible way to create a serial dense matrix D of dimension n by m is

```
#include "Epetra_SerialDenseMatrix.h"
Epetra_SerialDenseMatrix D(n,m);
```

One could also create a zero-size object,

```
Epetra_SerialDenseMatrix D();
```

and then shape this object:

```
D.Shape(n,m);
```

(`D` could be reshaped using `ReShape()`.)

An Epetra_SerialDenseMatrix is stored in a column-major order in the usual FORTRAN style. This class is built on the top of the BLAS library, and is derived from Epetra_Blas (not covered in this tutorial). Epetra_SerialDenseMatrix supports dense rectangular matrices.

To access the matrix element at the i-th row and the j-th column, it is possible to use the parenthesis operator (`A(i,j)`), or the bracket operator (`A[j][i]`, note that i and j are reversed)[5].

As an example of the use of this class, in the following code we consider a matrix-matrix product between two rectangular matrices `A` and `B`.

```
int NumRowsA = 2, NumColsA = 2;
int NumRowsB = 2, NumColsB = 1;
Epetra_SerialDenseMatrix A, B;
A.Shape(NumRowsA, NumColsA);
B.Shape(NumRowsB, NumColsB);
// ... here set the elements of A and B
Epetra_SerialDenseMatrix AtimesB;
AtimesB.Shape(NumRowsA,NumColsB);
double alpha = 1.0, beta = 1.0;
AtimesB.Multiply('N','N',alpha, A, B, beta);
cout << AtimesB;
```

`Multiply()` performs the operation $C = \alpha A + \beta B$, where A replaced by A^T if the first input parameter is `T`, and B replaced by B^T if the second input parameter is `T`. The corresponding source code file is `didasko/examples/epetra/ex10.cpp`.

To solve a linear system with a dense matrix, one has to create an Epetra_SerialDenseSolver. This class uses the most robust techniques available in the LAPACK library. The class is built on the top of BLAS and LAPACK, and thus has excellent performance and numerical stability[6].

The primary difference is that Epetra_LAPACK is a "thin" layer on the top of LAPACK, while Epetra_SerialDenseSolver attempts to provide easy access to the more robust dense linear solvers.

Epetra_LAPACK is preferable if the user seeks a convenient wrapper around the FORTRAN LAPACK routines, and the problem at hand is well-conditioned. Instead, when the user wants (or potentially wants to) solve ill-conditioned problems or favors a more object-oriented interface, then we suggest Epetra_SerialDenseMatrix..

Given an Epetra_SerialDenseMatrix and two Epetra_SerialDenseVectors `x` and `b`, the general approach is as follows:

---

[5]The bracket approach is in general faster, as the compiler can inline the corresponding function. Instead, some compiler have problems to inline the parenthesis operator.

[6]Another package, Teuchos, covered in Chapter 8, allows a templated access to LAPACK. ScaLAPACK is supported through Amesos, see Chapter 10.

```
Epetra_SerialDenseSolver Solver();
Solver.SetMatrix(D);
Solver.SetVectors(x,b);
```

Then, it is possible to invert the matrix with `Invert()`, solve the linear system with `Solve()`, apply iterative refinement with `ApplyRefinement()`. Other methods are available; for instance,

```
double rcond=Solve.RCOND();
```

returns the reciprocal of the condition number of matrix `D` (or -1 if not computed).

`didasko/examples/epetra/ex11.cpp` outlines some of the capabilities of the Epetra_SerialDenseSolver class.

3.2 Distributed Sparse Matrices

Epetra provides an extensive set of classes to create and fill distributed sparse matrices. These classes allow row-by-row or element-by-element constructions. Support is provided for common matrix operations, including scaling, norm, matrix-vector multiplication and matrix-multivector multiplication[7].

Using Epetra objects, applications do not need to know about the particular storage format, and other implementation details such as data layout, the number and location of ghost nodes. Epetra furnishes two basic formats, one suited for point matrices, the other for block matrices. The former is presented in this Section; the latter is introduced in Section 3.3. Other matrix formats can be introduced by deriving the Epetra_RowMatrix virtual class as needed.

Remark 8. *Some numerical algorithms require the application of the linear operator only. For this reason, some applications choose not to store a given matrix. Epetra can handle this situation using with the Epetra_Operator class; see Section 4.3.*

The process of creating a sparse matrix is more involved than the process for dense matrices. This is because, in order to obtain excellent numerical performance, one has to provide an estimation of the nonzero elements on each row of the sparse matrix. (Recall that dynamic allocation of new memory and copying the old storage into the new one is an expensive operation.)

As a general rule, the process of constructing a (distributed) sparse matrix is as follows:

- allocate an integer array `Nnz`, whose length equals the number of local rows;

- loop over the local rows, and estimate the number of nonzero elements of that row;

---

[7]Methods for matrix-matrix products are avaiable through the EpetraExt package. Another alternative is to use the efficient matrix-matrix product of ML, which requires ML_Operator objects. One may use light-weight conversions to ML_Operator, perform the ML matrix-matrix product, then convert the result to Epetra Matrix.

```
virtual int Multiply (bool TransA, const Epetra_MultiVector
&X, Epetra_MultiVector &Y) const=0
```
Returns the result of a Epetra_RowMatrix multiplied by a Epetra_MultiVector X in Y.

```
virtual int Solve (bool Upper, bool Trans,
bool UnitDiagonal, const Epetra_MultiVector &X,
Epetra_MultiVector &Y) const=0
```
Returns result of a local-only solve using a triangular Epetra_RowMatrix with Epetra_MultiVectors X and Y.

```
virtual int InvRowSums (Epetra_Vector &x) const=0
```
Computes the sum of absolute values of the rows of the Epetra_RowMatrix, results returned in x.

```
virtual int LeftScale (const Epetra_Vector &x)=0
```
Scales the Epetra_RowMatrix on the left with a Epetra_Vector x.

```
virtual int InvColSums (Epetra_Vector &x) const=0
```
Computes the sum of absolute values of the cols of the Epetra_RowMatrix, results returned in x.

```
virtual int RightScale (const Epetra_Vector &x)=0
```
Scales the Epetra_RowMatrix on the right with a Epetra_Vector x.

Table 4. Mathematical methods of Epetra_RowMatrix

```
virtual bool Filled () const=0
```
If FillComplete() has been called, this query returns true, otherwise it returns false.
```
virtual double NormInf () const=0
```
Returns the infinity norm of the global matrix.
```
virtual double NormOne () const=0
```
Returns the one norm of the global matrix.
```
virtual int NumGlobalNonzeros () const=0
```
Returns the number of nonzero entries in the global matrix.
```
virtual int NumGlobalRows () const=0
```
Returns the number of global matrix rows.
```
virtual int NumGlobalCols () const=0
```
Returns the number of global matrix columns.
```
virtual int NumGlobalDiagonals () const=0
```
Returns the number of global nonzero diagonal entries, based on global row/column index comparisons.
```
virtual int NumMyNonzeros () const=0
```
Returns the number of nonzero entries in the calling processor's portion of the matrix.
```
virtual int NumMyRows () const=0
```
Returns the number of matrix rows owned by the calling processor.
```
virtual int NumMyCols () const=0
```
Returns the number of matrix columns owned by the calling processor.
```
virtual int NumMyDiagonals () const=0
```
Returns the number of local nonzero diagonal entries, based on global row/column index comparisons.
```
virtual bool LowerTriangular () const=0
```
If matrix is lower triangular in local index space, this query returns true, otherwise it returns false.
```
virtual bool UpperTriangular () const=0
```
If matrix is upper triangular in local index space, this query returns true, otherwise it returns false.
```
virtual const Epetra_Map & RowMatrixRowMap () const=0
```
Returns the Epetra_Map object associated with the rows of this matrix.
```
virtual const Epetra_Map & RowMatrixColMap () const=0
```
Returns the Epetra_Map object associated with the columns of this matrix.
```
virtual const Epetra_Import * RowMatrixImporter () const=0
```
Returns the Epetra_Import object that contains the import operations for distributed operations.

Table 5. Atribute access methods of `Epetra_RowMatrix`

- create the sparse matrix using `Nnz`;

- fill the sparse matrix.

As an example, in this Section we will present how to construct a distributed (sparse) matrix, arising from a finite-difference solution of a one-dimensional Laplace problem. This matrix looks like:

$$A = \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \cdots & \cdots & \cdots & -1 \\ & & & -1 & 2 \end{pmatrix}.$$

The example illustrates how to construct the matrix, and how to perform matrix-vector operations. The code can be found in `didasko/examples/epetra/ex12.cpp`.

We start by specifying the global dimension (here is 5, but can be any number):

```
int NumGlobalElements = 5;
```

We create a map (for the sake of simplicity linear), and define the local number of rows and the global numbering for each local row:

```
Epetra_Map Map(NumGlobalElements,0,Comm);
int NumMyElements = Map.NumMyElements();
int * MyGlobalElements = Map.MyGlobalElements( );
```

In particular, we have that `j=MyGlobalElements[i]` is the global numbering for local node `i`. Then, we have to specify the number of nonzeros per row. In general, this can be done in two ways:

- Furnish an integer value, representing the number of nonzero element on each row (the same value for all the rows);

- Furnish an integer vector `NumNz`, of length `NumMyElements()`, containing the nonzero elements of each row.

The first approach is trivial: the matrix is create with the simple instruction

```
Epetra_CrsMatrix A(Copy,Map,3);
```

(The `Copy` keyword is explained in Section 2.4.) In this case, Epetra considers the number 3 as a "suggestion," in the sense that the user can still add more than 3 elements per row (at the price of a possible performance decay). The second approach is as follows:

```
int * NumNz = new int[NumMyElements];
for( int i=0 ; i<NumMyElements ; i++ )
if( MyGlobalElements[i]==0 ||
```

```
    MyGlobalElements[i] == NumGlobalElements-1)
  NumNz[i] = 2;
else
  NumNz[i] = 3;
```

We are building a tridiagonal matrix where each row has (-1 2 -1). Here `NumNz[i]` is the number of nonzero terms in the i-th global equation on this process (2 off-diagonal terms, except for the first and last equation).

   Now, the command to create an Epetra_CsrMatrix is

```
Epetra_CrsMatrix A(Copy,Map,NumNz);
```

We add rows one at a time. The matrix `A` has been created in Copy mode, in a way that relies on the specified map. To fill its values, we need some additional variables: let us call them `Indexes` and `Values`. For each row, `Indices` contains global column indices, and `Values` the correspondingly values.

```
double * Values = new double[2];
Values[0] = -1.0; Values[1] = -1.0;
int * Indices = new int[2];
double two = 2.0;
int NumEntries;

for( int i=0 ; i<NumMyElements; ++i ) {
  if (MyGlobalElements[i]==0) {
      Indices[0] = 1;
      NumEntries = 1;
  } else if (MyGlobalElements[i] == NumGlobalElements-1) {
    Indices[0] = NumGlobalElements-2;
    NumEntries = 1;
  } else {
    Indices[0] = MyGlobalElements[i]-1;
    Indices[1] = MyGlobalElements[i]+1;
    NumEntries = 2;
  }
  A.InsertGlobalValues(MyGlobalElements[i], NumEntries,
                       Values, Indices);
  // Put in the diagonal entry
  A.InsertGlobalValues(MyGlobalElements[i], 1, &two,
                       MyGlobalElements+i);
}
```

Note that column indices have been inserted using global indices (but a method called `InsertMyValues` can be used as well) . Finally, we transform the matrix representation into one based on local

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indexes. The transformation in required in order to perform efficient parallel matrix-vector products and other matrix operations.

```
A.FillComplete();
```

This call to `FillComplete()` will reorganize the internally stored data so that each process knows the set of internal, border and external elements for a matrix-vector product of the form $B = AX$. Also, the communication pattern is established. As we have specified just one map, Epetra considers that the the rows of A are distributed among the processes in the same way of the elements of X and B. Although standard, this approach is only a particular case. Epetra allows the user to handle the more general case of a matrix whose Map differs from that of X and that of B. In fact, each Epetra matrix is defined by *four* maps:

- Two maps, called RowMap and ColumnMap, define the sets of rows and columns of the elements assigned to a given processor. In general, one processor cannot set elements assigned to other processors[8]. RowMap and ColumnMap define the pattern of the matrix, as it is used during the construction. They can be obtained using the methods `RowMatrixRowMap()` and `RowMatrixColMap()` of the Epetra_RowMatrix class. Usually, as a ColumnMap is not specified, it is automatically created by Epetra. In general RowMap and ColumnMap can differ.

- DomainMap and RangeMap define, instead, the parallel data layout of X and B, respectively. Note that those two maps can be completely different from RowMap and ColumnMap, meaning that a matrix can be constructed using a certain data distribution, then used on vectors with another data distribution. DomainMap and RangeMap can differ. Maps can be obtained using the methods `DomainMap()` and `RangeMap()`.

The potential of the approach are illustated by the example file `didasko/examples/epetra/ex24.cpp`. In this example, to be run using two processors, we build two maps: `MapA` will be used to construct the matrix, while `MapB` to define the parallel layout of the vectors X and B. For the sake of simplicity, A is diagonal.

```
Epetra_CrsMatrix A(Copy,MapA,MapA,1);
```

As usual in this Tutorial, the integer vector `MyGlobalElementsA` contains the global ID of local nodes. To assemble A, we cycle over all the local rows (defined by `MapA`):

```
for( int i=0 ; i<NumElementsA ; ++i ) {
  double one = 2.0;
  int indices = MyGlobalElementsA[i];
  A.InsertGlobalValues(MyGlobalElementsA[i], 1, &one, &indices );
}
```

---

[8]Some classes, derived from the Epetra_RowMatrix, can perform data exchange; see for instance Epetra_FECrsMatrix or Epetra_FEVbrMatrix.

Now, as both X and B are defined using `MapB`, instead of calling `FillComplete()`, invoke

`A.FillComplete(MapB,MapB);`

Now, we can create X and B as vectors based on `MapB`, and perform the matrix-vector product:

```
Epetra_Vector X(MapB);   Epetra_Vector B(MapB);
A.Multiply(false,X,B);
```

Remark 9. *Although presented for Epetra_CrsMatrix objects, the distinction between RowMap, ColMap, DomainMap, and RangeMap holds for all classed derived from Epetra_RowMatrix.*

Example `didasko/examples/epetra/ex14.cpp` shows the use of some of the methods of the Epetra_CrsMatrix class. The code prints out information about the structure of the matrix and its properties. The output will be approximatively as reported here:

```
[msala:epetra]> mpirun -np 2 ./ex14
*** general Information about the matrix
Number of Global Rows = 5
Number of Global Cols = 5
is the matrix square  = yes
||A||_\infty          = 4
||A||_1               = 4
||A||_F               = 5.2915
Number of nonzero diagonal entries = 5( 100 %)
Nonzero per row : min = 2 average = 2.6 max = 3
Maximum number of nonzero elements/row = 3
min( a_{i,j} )       = -1
max( a_{i,j} )       = 2
min( abs(a_{i,j}) ) = 1
max( abs(a_{i,j}) ) = 2
Number of diagonal dominant rows        = 2 (40 % of total)
Number of weakly diagonal dominant rows = 3 (60 % of total)
*** Information about the Trilinos storage
Base Index              = 0
is storage optimized    = no
are indices global      = no
is matrix lower triangular = no
is matrix upper triangular = no
are there diagonal entries = yes
is matrix sorted        = yes
```

Other examples for Epetra_CrsMatrix include:

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- Example `didasko/examples/epetra/ex13.cpp` implements a simple distributed finite-element solver. The code solves a 2D Laplace problem with unstructured triangular grids. In this example, the information about the grid is hardwired. The interested user can easily modify those lines in order to read the grid information from a file.

- Example `didasko/examples/epetra/ex15.cpp` explains how to export an Epetra_CrsMatrix to file in a MATLAB format. The output of this example will be approximatively as follows:

```
[msala:epetra]> mpirun -np 2 ./ex15
A = spalloc(5,5,13);
% On proc 0: 3 rows and 8 nonzeros
A(1,1) = 2;
A(1,2) = -1;
A(2,1) = -1;
A(2,2) = 2;
A(2,3) = -1;
A(3,2) = -1;
A(3,3) = 2;
A(3,4) = -1;
% On proc 1: 2 rows and 5 nonzeros
A(4,4) = 2;
A(4,5) = -1;
A(4,3) = -1;
A(5,4) = -1;
A(5,5) = 2;
```

  A companion to this example is `didasko/examples/epetra/ex16.cpp`, which exports an Epetra_Vector to MATLAB format. Note also that the package EpetraExt contains several purpose tools to read and write matrices in various formats.

3.3 Creating Block Matrices

This section reviews how to work with block matrices (where each block is a dense matrix)[9]. This format has been designed for PDE problems with more than one unknown per grid node. The resulting matrix has a sparse block structure, and the size of each dense block equals the number of PDE equations defined on that block. This format is quite general, and can handle matrices with variable block size, as is done is the following example.

---

[9]Trilinos offers capabilities to deal with matrices composed by few sparse blocks, like for instance matrices arising from the discretization of the incompressible Navier-Stokes equations, through the Meros package (not covered in this tutorial).

First, we create a map, containing the distribution of the blocks:

```
Epetra_Map Map(NumGlobalElements,0,Comm);
```

Here, a linear decomposition is used for the sake of simplicity, but any map may be used as well. Now, we obtain some information about the map:

```
// local number of elements
int NumMyElements = Map.NumMyElements();
// global numbering of local elements
int * MyGlobalElements = new int [NumMyElements];
Map.MyGlobalElements( MyGlobalElements );
```

A block matrix can have blocks of different size. Here, we suppose that the dimension of diagonal block row i is $i + 1$. The integer vector `ElementSizeList` will contain the block size of local element `i`.

```
Epetra_IntSerialDenseVector ElementSizeList(NumMyElements);
 for( int i=0 ; i<NumMyElements ; ++i )
   ElementSizeList[i] = 1+MyGlobalElements[i];
```

Here `ElementSizeList` is declared as Epetra_IntSerialDenseVector, but an int array is fine as well.

Now we can create a map for the block distribution:

```
Epetra_BlockMap BlockMap(NumGlobalElements,NumMyElements,
                         MyGlobalElements,
                         ElementSizeList.Values(),0,Comm);
```

and finally we can create the VBR matrix based on `BlockMap`. In this case, nonzero elements are located in the diagonal and the sub-diagonal above the diagonal.

```
Epetra_VbrMatrix A(Copy, BlockMap, 2);

int Indices[2];
double Values[MaxBlockSize];

for( int i=0 ; i<NumMyElements ; ++i ) {
  int GlobalNode = MyGlobalElements[i];
  Indices[0] = GlobalNode;
  int NumEntries = 1;
  if( GlobalNode != NumGlobalElements-1 ) {
    Indices[1] = GlobalNode+1;
    NumEntries++;
```

```
  }
  A.BeginInsertGlobalValues(GlobalNode, NumEntries, Indices);
  // insert diagonal
  int BlockRows = ElementSizeList[i];
  for( int k=0 ; k<BlockRows * BlockRows ; ++k )
    Values[k] = 1.0*i;
  B.SubmitBlockEntry(Values,BlockRows,BlockRows,BlockRows);

  // insert off diagonal if any
  if( GlobalNode != NumGlobalElements-1 ) {
    int BlockCols = ElementSizeList[i+1];
    for( int k=0 ; k<BlockRows * BlockCols ; ++k )
      Values[k] = 1.0*i;
    B.SubmitBlockEntry(Values,BlockRows,BlockRows,BlockCols);
  }
  B.EndSubmitEntries();
}
```

Note that, with VBR matrices, we have to insert one block at time. This required two more instructions, one to start this process (`BeginInsertGlobalValues`), and another one to commit the end of submissions (`EndSubmitEntries`). Similar functions to sum and replace elements exist as well.

Please refer to `didasko/examples/epetra/ex17.cpp` for the entire source.

3.4   Insert non-local Elements Using FE Matrices

The most important additional feature provided by the Epetra_FECrsMatrix with respect to Epetra_CrsMatrix, is the capability to set non-local matrix elements. We will illustrate this using the following example, reported in
`didasko/examples/epetra/ex23.cpp`. In the example, we will set all the entries of a distributed matrix from process 0. For the sake of simplicity, this matrix is diagonal, but more complex cases can be handled as well.

First, we define the Epetra_FECrsMatrix in Copy mode as

```
Epetra_FECrsMatrix A(Copy,Map,1);
```

Now, we will set all the diagonal entries from process 0:

```
if( Comm.MyPID() == 0 ) {
  for( int i=0 ; i<NumGlobalElements ; ++i ) {
    int indices[2];
    indices[0] = i; indices[1] = i;
```

```
        double value = 1.0*i;
        A.SumIntoGlobalValues(1,indices,&value);
    }
}
```

The Function `SumIntoGlobalValues` adds the coefficients specified in `indices` (as pair row-column) to the matrix, adding them to any coefficient that may exist at the specified location. In a finite element code, the user will probably insert more than one coefficient at time (typically, all the matrix entries corresponding to an elemental matrix).

Next, we need to exchange data, to that each matrix element not owned by process 0 could be send to the owner, as specified by `Map`. This is accomplished by calling, on all processes,

```
A.GlobalAssemble();
```

A simple

```
cout << A;
```

can be used to verify the data exchange.

4 Other Epetra Classes

Epetra includes classes that facilitate the development of parallel codes. In this Chapter we will recall the main usage of some of those classes:

- Epetra_Time (in Section 4.1);

- Epetra_Flops (in Section 4.2).

- Epetra_Operator and Epetra_RowMatrix (in Section 4.3);

- Epetra_LinearProblem (in Section 4.4).

4.1 Epetra_Time

Retrieving elapsed and wall-clock time is problematic due to platform-dependent and language-dependent issues. To avoid those problems, Epetra furnishes the Epetra_Time class. Epetra_Time is meant to insulate the user from the specifics of timing among a variety of platforms. Using Epetra_Time, it is possible to measure the elapsed time. This is the time elapsed between two phases of a program.

An Epetra_Time object is defined as

```
Epetra_Time time(Comm);
```

(`Comm` being an Epetra_Comm object, see Section 2.1.) To compute the elapsed time required by a given piece of code, then user should put the instruction

```
time.ResetStartTime();
```

before the code to be timed. `ElapsedTime()` returns the elapsed time from the creation of *this* object or from the last call to `ResetStartTime()`.

4.2 Epetra_Flops

The Epetra_Flops class provides basic support and consistent interfaces for counting and reporting floating point operations performed in the Epetra computational classes. All classes based on the Epetra_CompObject can count flops by the user creating an Epetra_Flops object and calling the SetFlopCounter() method for an Epetra_CompObject.

As an example, suppose you are interested in counting the flops required by a vector-vector product (between, say, `x` and `y`). The first step is to create an instance of the class:

```
Epetra_Flops counter();
```

Then, it is necessary to "hook" the counter object to the desired computational object, in the following way:

```
x.SetFlopCounter(counter);
y.SetFlopCounter(counter);
```

Then, perform the desired computations on Epetra objects, like

```
x.Dot(y,&dotProduct);
```

Finally, extract the number of performed operations and stored it in the double variable `total_flops` as

```
total_flops = counter.Flops();
```

This returns the total number of *serial* flops, and then resets the flop counter to zero.

Epetra_Time objects can be used in conjunction with Epetra_Flops objects to estimate the number of floating point operations per second of a given code (or a part of it). One can proceed as here reported:

```
Epetra_Flops counter;
x.SetFlopCounter(counter);
Epetra_Time timer(Comm);
x.Dot(y,&dotProduct);
double elapsed_time = timer.ElapsedTime();
double total_flops =counter.Flops();
cout << "Total ops: " << total_flops << endl;
double MFLOPs = total_flops/elapsed_time/1000000.0;
cout << "Total MFLOPs  for mat-vec = " << MFLOPs << endl<< endl;
```

This code is reported in `didasko/examples/epetra/ex20.cpp`. The output will be approximatively as follows:

```
[msala:epetra]> mpirun -np 2 ./ex20
Total ops: 734
Total MFLOPs  for mat-vec = 6.92688

Total ops: 734
Total MFLOPs  for mat-vec = 2.48021

Total ops: 246
Total MFLOPs for vec-vec = 0.500985

q dot z = 2
Total ops: 246
Total MFLOPs for vec-vec = 0.592825

q dot z = 2
```

Remark 10. *Operation count are serial count, and therefore keep track of local operations only.*

Remark 11. *Each computational class has a* `Flops()` *method, that may be queried for the flop count of that object.*

4.3 Epetra_Operator and Epetra_RowMatrix Classes

Matrix-free methods are introduced in the Epetra framework using either of the following two classes:

- Epetra_Operator;

- Epetra_RowMatrix.

Each class is a pure virtual class (specifing interfaces only), that enable the use of real-valued double-precision sparse matrices. Epetra_RowMatrix, derived from Epetra_Operator, is meant for matrices where the matrix entries are intended for row access, and it is currently implemented by Epetra_CrsMatrix, Epetra_VbrMatrix, Epetra_FECrsMatrix, and Epetra_FEVbrMatrix.

Consider for example the 3-point centered difference discretization of a one dimensional Laplacian on a regular grid. For the sake of simplicity, we avoid the issues related to intra-process communication (hence this code can be run with one process only).

The first step is the definition of a class, here called `TriDiagonalOperator`, and derived from the Epetra_Operator class.

```
class TriDiagonalOperator : public Epetra_Operator {
public:
 // .. definitions here, constructors and methods
private:
  Epetra_Map Map_;
  double diag_minus_one_;   // value in the sub-diagonal
  double diag_;             // value in the diagonal
  double diag_plus_one_;    // value in the super-diagonal
}
```

As the class Epetra_Operator implements several virtual methods, we have to specify all those methods in our class. Among them, we are interested in the `Apply` method, which may be coded as follows:

```
int Apply( const Epetra_MultiVector & X,
           Epetra_MultiVector & Y ) const {
  int Length = X.MyLength();
```

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```
  // need to handle multi-vectors and not only vectors
  for( int vec=0 ; vec<X.NumVectors() ; ++vec ) {

    // one-dimensional problems here
    if( Length == 1 ) {
      Y[vec][0] = diag_ * X[vec][0];
      break;
    }

    // more general case (Lenght >= 2)
    // first row
    Y[vec][0] = diag_ * X[vec][0] + diag_plus_one_ * X[vec][1];

    // intermediate rows
    for( int i=1 ; i<Length-1 ; ++i ) {
      Y[vec][i] = diag_ * X[vec][i] + diag_plus_one_ * X[vec][i+1]
        + diag_minus_one_ * X[vec][i-1];
    }
    // final row
    Y[vec][Length-1] = diag_ * X[vec][Length-1]
      + diag_minus_one_ * X[vec][Length-2];
  }
  return true;
}
```

Now, in the `main` function, we can define a TriDiagonalOperatr object using the specified constructor:

```
  TriDiagonalOperator TriDiagOp(-1.0,2.0,-1.0,Map);
```

and

```
  DiagOp.Apply(x,y);
```

computes the discrete Laplacian on `x` and returns the product in `y`.
`didasko/examples/epetra/ex21.cpp` reportes the complete source code.

Remark 12. *The clear disadvantage of deriving Epetra_Operator or Epetra_RowMatrix with respect to use Epetra_CrsMatrix or Epetra_VbrMatrix, is that users must specify their communication pattern for intra-process data exchange. For this purpose, Epetra_Import classes can be used. File* `didasko/examples/epetra/ex22.cpp` *shows how to extend* `ex21.cpp` *to the multi-process case. This example makes use of the Epetra_Import class to exchange data.*

Another use of Epetra_Operator and Epetra_RowMatrix is to allow support for user defined matrix format. For instance, suppose that your code generates matrices in MSR format (detailed in the Aztec documentation). You can easily create an Epetra_Operator, that applies the MSR format to Epetra_MultiVectors. For the sake of simplicity, we will limit ourselves to the serial case. In the distributed, we must also handle ghost-node updates.

As a first step, we create a class, derived from the Epetra_Operator class,

```cpp
class MSRMatrix : public Epetra_Operator
{

public:
  // constructor
  MSRMatrix(Epetra_Map Map, int * bindx, double * val) :
    Map_(Map), bindx_(bindx), val_(val)
  {}

  ~MSRMatrix() // destructor
  {}

  // Apply the RowMatrix to a MultiVector
  int Apply(const Epetra_MultiVector & X, Epetra_MultiVector & Y ) const
  {

    int Nrows = bindx_[0]-1;

    for( int i=0 ; i<Nrows ; i++ ) {
      // diagonal element
      for( int vec=0 ; vec<X.NumVectors() ; ++vec ) {
        Y[vec][i] = val_[i]*X[vec][i];
      }
      // off-diagonal elements
      for( int j=bindx_[i] ; j<bindx_[i+1] ; j++ ) {
        for( int vec=0 ; vec<X.NumVectors() ; ++vec ) {
          Y[vec][bindx_[j]] += val_[j]*X[vec][bindx_[j]];
        }
      }
    }
    return 0;
  } /* Apply */
  ... other functions ...

private:
```

```
   int * bindx_; double * val_;
}
```

In this sketch of code, the constructor takes the two MSR vectors, and an Epetra_Map. The complete code is reported in
```
didasko/examples/epetra/ex25.cpp.
```

4.4 Epetra_LinearProblem

A linear system $AX = B$ is defined by an Epetra_LinearProblem class. The class requires an Epetra_RowMatrix or an Epetra_Operator object (often an Epetra_CrsMatrix or Epetra_VbrMatrix), and two (multi-)vectors X and B. X must have been defined using a map equivalent to the DomainMap of A, while B using a map equivalent ot the RangeMap of A (see Section 3.2).

Linear systems may be solved either by iterative methods (typically, using AztecOO, covered in Chapter 6), or by direct solvers (typically, using Amesos, described in Chapter 10).

Once the linear problem has been defined, the user can:

- scale the problem, using `LeftScale(D)` or `RightScale(D)`, D being an Epetra_Vector of compatible size;

- change X and B, using `SetRHS(&B)` and `SetLHS(&X)`;

- change A, using `SetOperator(&A)`.

Please refer to Table 6 for a summary of the methods.

4.5 Concluding Remarks on Epetra

More details about the Epetra project, and a technical description of classes and methods, can be found in [15, 19].

```
void SetPDL (ProblemDifficultyLevel PDL)
```
Set problem difficulty level.
```
void SetOperator (Epetra_RowMatrix *A)
```
Set Operator A of linear problem AX = B using an Epetra_RowMatrix.
```
void SetOperator (Epetra_Operator *A)
```
Set Operator A of linear problem AX = B using an Epetra_Operator.
```
void SetLHS (Epetra_MultiVector *X)
```
Set left-hand side X of linear problem AX = B.
```
void SetRHS (Epetra_MultiVector *B)
```
Set right-hand side B of linear problem AX = B.
```
int CheckInput () const
```
Check input parameters for existence and size consistency.
```
int LeftScale (const Epetra_Vector &D)
```
Perform left scaling of a linear problem.
```
int RightScale (const Epetra_Vector &D)
```
Perform right scaling of a linear problem.
```
Epetra_Operator * GetOperator () const
```
Get a pointer to the operator A.
```
Epetra_RowMatrix * GetMatrix () const
```
Get a pointer to the matrix A.
```
Epetra_MultiVector * GetLHS () const
```
Get a pointer to the left-hand-side X.
```
Epetra_MultiVector * GetRHS () const
```
Get a pointer to the right-hand-side B.
```
ProblemDifficultyLevel GetPDL () const
```
Get problem difficulty level.
```
bool IsOperatorSymmetric () const
```
Get operator symmetry bool.

Table 6. Methods of Epetra_LinearProblem

5   Generating Linear Systems with Triutils

This Chapter presents two functionalities of Triutils, that will be extensively used in the examples of the later chapters:

- the Triutils command line parser (in Section 5.1);

- the Triutils matrix generator (in Section 5.2).

Some readers may choose to skip this Chapter because their application is their example. However, it does help to find the test matrices closest to the ones in their code for several reasons. Using well-chosen matrices, simple but sufficiently close to the final application, one can quickly test the performances of a given set of algorithms on the problem of interest, using a serial or a parallel environment. Several test matrices exhibit a known behavior (e.g., theory predicts the behavior of the condition number), and can be used to validate algorithms. Besides, as they can be quickly generate with few code lines, experts may use them to optimize or fix thier code. Therefore, a short code using a gallery matrix may be used to communicate with developers.

5.1   Trilinos_Util::CommandLineParser

It is possible to use the
`Trilinos_Util::CommandLineParser` class to parse the command line. With this class, it is easy to handle input line arguments and shell-defined variables. For instance, the user can write

```
[msala:triutils]>ex2.exe -nx 10 -tol 1e-6 -solver=cg -output
```

and, in the code, easily obtain the value of `nx`, `tol`, and `solver`, using a simple code as follows:

```
int main(int argc, char *argv[])
{

  Trilinos_Util::CommandLineParser CLP(argc,argv);
  int nx = CLP.Get("-nx", 123);
  int ny = CLP.Get("-ny", 145);
  double tol = CLP.Get("-tol", 1e-12);
  string solver = CLP.Get("-solver","gmres");

  bool Output = CLP.Has("-output");

  cout << "nx = " << nx << endl;
```

```
  cout << "ny = " << ny << " (default value)" << endl;
  cout << "tol = " << tol << endl;
  cout << "solver = " << solver << endl;

  return 0;
}
```

In the command line, the user can specify a value for a given option in the following ways:

- `-tolerance 1e-12` (with one or more spaces)

- `-tolerance=1e-12` (with = sign and no spaces)

Option names must begin with one or more dashes ('-'). Options may have at most one value.

If option name is not found in the database, the default value is returned. If needed, the user can also specify a default value to return when the option name is not found in the database. The method `HaveOption` will query the database for an option.

File `didasko/examples/triutils/ex2.cpp` gives an example of the usage of this class.

5.2   Trilinos_Util::CrsMatrixGallery

The class `Trilinos_Util::CrsMatrixGallery` provides functions similar to the MAT-LAB's `gallery` function[10].

A typical constructor requires the problem type and an Epetra_Comm, and is followed by a set of instructions to specify the problem. The following example creates a matrix corresponding to the discretization of a 2D Laplacian on a Cartesian grid with 100 points:

```
Trilinos_Util::CrsMatrixGallery Gallery("laplace_2d", Comm);
Gallery.Set("problem_size",100);
Gallery.Set("map_type","linear");
Gallery.Set("exact_solution","random");
```

The nodes are decomposed linearly, and the exact solution is a random vector.

The next example will read a matrix stored in Harwell/Boeing format:

```
Trilinos_Util::CrsMatrixGallery Gallery("hb", Comm);
Gallery.Set("matrix_name","bcsstk14.rsa");
Gallery.Set("map_type","greedy");
```

---

[10]Many of the matrices that can be created using Trilinos_Util::CrsMatrixGallery are equivalent or simiilar to those provided by the MATLAB© function `gallery`. In these cases, the reader is referred to the MATLAB documentation for more details about the matrices' properties.

The example reads the matrix (and, if available, solution and right-hand side) from the file `bcsstk14.rsa`, and partitions the matrix across the processes using a simple greedy algorithm.

Once all the required parameters have been specified, the user can get a pointer to the constructed Epetra_CrsMatrix, to the exact and starting solution, and to the right-hand side (both Epetra_Vector's):

```
A = Gallery.GetMatrix();
ExactSolution = Gallery.GetExactSolution();
RHS = Gallery.GetRHS();
StartingSolution = Gallery.GetStartingSolution();
```

An Epetra_LinearProblem is defined by

```
Epetra_LinearProblem Problem(A,StartingSolution,RHS);
```

Next one may use AztecOO to solve the linear system:

```
AztecOO Solver(Problem);
Solver.SetAztecOption( AZ_precond, AZ_dom_decomp );
Solver.Iterate(1000,1E-9);
```

Using Trilinos_Util::MatrixGallery, one computes the true residual and the difference between computed and exact solution:

```
double residual;
Gallery.ComputeResidual(&residual);

Gallery.ComputeDiffBetweenStartingAndExactSolutions(&residual);
```

A list of methods implemented by Trilinos_Util::CrsMatrixGallery is reported in Table 7.
The matrix can be written on a file in MATLAB format, using

```
string FileName = "matrix.m";
bool UseSparse = false;
Gallert.WriteMatrix(FileName,UseSparse);
```

If `UseSparse` is true, the matrix is created in sparse format (using the MATLAB command `spalloc`).

To sum up, the main options reviewed here are:

 problem_type   [string] Specifies the problem type. A list of currently available problems is reported later in this section.

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```
GetMatrix()
```
Returns a pointer to the internally stored Epetra_CrsMatrix.

```
GetExactSolution()
```
Returns a pointer to the internally stored exact solution vector (as an Epetra_Vector).

```
GetStartingSolution()
```
Returns a pointer to the internally stored starting solution vector (as an Epetra_Vector).

```
GetRhs()
```
Returns a pointer to the internally stored right-hand side (as an Epetra_Vector).

```
GetLinearProblem()
```
Returns a pointer to the internally stored Epetra_LinearProblem for the VBR matrix).

```
GetMap()
```
Returns a pointer to the internally stored Epetra_Map.

Table 7. Methods of class Trilinos_Util::CrsMatrixGallery.

problem_size	[int] Size of the problem. Note that some problems, defined on structured meshes, allow the specification of the number of nodes on the x-, y- and z-axis. Please refer to each problem's description for more details.
nx	[int] Number of nodes in the x-direction (if supported by the specific problem).
ny	[int] Number of nodes in the y-direction (if supported by the specific problem).
nz	[int] Number of nodes in the z-direction (if supported by the specific problem).
mx	[int] Number of processes in the x-direction (if supported by the specific problem).
my	[int] Number of processes in the y-direction (if supported by the specific problem).
mz	[int] Number of processes in the z-direction (if supported by the specific problem).
map_type	[string] Defines the data layout across the processes. See Table 8.
exact_solution	[string] Defines the exact solution. See Table 9.
starting_solution	[string] Defines the starting solution vector. It can be: zero or random.

A list of currently avaialble problems is reported below. We use the following notation. IntValue always refer to a generic positive integer. The following symbols a,b,c,d,e,f,g always refer to double-precision values. Note that some matrices are dense, but still stored as Epetra_CrsMatrix, a sparse matrix format. The generic (i, j) element of a given matrix is $A_{i,j}$ (for simplicity, we suppose that indices start from 1[11]). n represents the matrix size.

eye	Creates an identity matrix. The size of the problem is set using Set("problem size", IntValue), or, alternatively, by Set("nx", IntValue).

---

[11]It is understood that, in the actual implementation, indices start from 0.

`linear`	Create a linear map. Elements are divided into continuous chunks among the processors. This is the default value.
`box`	Used for problems defined on Cartesian grids over a square. The domain is subdivided into `mx x my` subdomains. `mx` and `my` are automatically computed if the total number or processes is a perfect square. Alternatively, `mx` and `my` are specified via `Set("mx",IntValue)` and `Set("my",IntValue)`.
`interlaced`	Elements are subdivided so that element i is assigned to process i%NumProcs.
`random`	Assign each node to a random process
`greedy`	(only for HB matrices) implements a greedy algorithm to decompose the graph of the HB matrix among the processes

Table 8. Available `map_type` optins.

`random`	Random values
`constant`	All elements set to 1.0.
`quad_x`	Nodes are supposed to be distributed over the 1D segment $(0, 1)$, with equal spacing, and the solution is computed as $x(1 - x)$.
`quad_xy`	Nodes are supposed to be distributed over the square $(0, 1) \times (0, 1)$, with equal spacing, and the solution is computed as $x(1 - x)y(1 - y)$.

Table 9. Available `exact_solution` options.

cauchy	Creates a particular instance of a Cauchy matrix with elements $A_{i,j} = 1/(i + j)$. Explicit formulas are known for the inverse and determinant of a Cauchy matrix. For this particular Cauchy matrix, the determinant is nonzero and the matrix is totally positive.
cross_stencil_2d	Creates a matrix with the same stencil of laplace_2d, but with arbitrary values. The stencil is $$A = \begin{bmatrix} & e & \\ b & a & c \\ & d & \end{bmatrix}.$$ The default values are a=5, b=c=d=e=1. The problem size is specified as in laplace_2d.
cross_stencil_3d	Similar to the 2D case. The matrix stencil correspond to that of a 3D Laplace operator on a structured grid. On a given x-y plane, the stencil is as in laplace 2d. The value on the plane below is set using Set("f",F), and in the plane above with Set("g",G"). The default values are a=7,b=c=d=e=f=g=1. The problem size is specified as in laplace3d.
diag	Creates a diagonal matrix. The elements on the diagonal can be set using Set("a",value). Default value is a = 1. The problem size is set as for eye.
fiedler	Creates a matrix whose element are $A_{i,j} = |i - j|$. The matrix is symmetric, and has a dominant positive eigenvalue, and all the other eigenvalues are negative.
hanowa	Creates a matrix whose eigenvalues lie on a vertical line in the complex plane. The matrix has the 2x2 block structure (in MATLAB's notation) $$A = \begin{bmatrix} a * \text{eye}(n/2) & -\text{diag}(1{:}m) \\ \text{diag}(1{:}m) & a * \text{eye}(n/2) \end{bmatrix}.$$ The complex eigenvalues are of the form $a\,k\sqrt{-1}$ and $-k\sqrt{-1}$, for $1 \le k \le n/2$. The default value for a is -1.
hb	The matrix is read from file. File name is specified by Set("file name", FileName). FileName is a C++ string. The problem size is automatically determined.

`hilbert`	This is a famous example of a badly conditioned matrix. The elements are defined as $A_{i,j} = 1/(i + j)$.
`jordblock`	Creates a Jordan block with eigenvalue set via `Set("a",DoubleVal)`. The default value is 0.1. The problem size is specified as for `eye`.
`kms`	Create the $n \times n$ Kac-Murdock-Szegö Toepliz matrix such that $A_{i,j} = \rho^{\|i-j\|}$ (for real ρ only). Default value is $\rho = 0.5$, or can be set as `Set("a",value)`. The inverse of this matrix is tridiagonal, and the matrix is positive definite if and only if $0 < \|\rho\| < 1$.
`laplace_1d`	Creates the classical tridiagonal matrix with stencil [-1, 2, -1]. The problem size is specified as for `eye`.
`laplace_1d_n`	As for `laplace_1d`, but with Neumann boundary condition. The matrix is singular.
`laplace_2d`	Creates a matrix corresponding to the stencil of a 2D Laplacian operator on a structured Cartesian grid. The problem size is specified using `Set("problem size", IntValue)`. In this case, IntValue must be a perfect square. Alternatively, one can set the number of nodes along the x-axis and y-axis, using `Set("nx",IntValue)` and `Set("ny",IntValue)`.
`laplace_2d_n`	As for `laplace_2d`, but with Neumann boundary condition. The matrix is singular.
`laplace_3d`	Creates a matrix corresponding to the stencil of a 3D Laplacian operator on a structured Cartesian grid. The problem size is specified using `Set("problem size",IntValue)`. In this case, IntValue must be a cube. Alternatively, one can specify the number of nodes along the axis, using `Set("nx",IntValue)`, `Set("ny",IntValue)`, and `Set("nz",IntValue)`.

lehmer	Returns a symmetric positive definite matrix, such that

$$A_{i,j} = \begin{cases} \frac{i}{j} & \text{if } j \geq i \\ \frac{j}{i} & \text{otherwise} \end{cases}.$$

This matrix has three properties: is totally nonnegative, the inverse is tridiagonal and explicitly known, The condition number is bounded as $n \leq cond(A) \leq 4 * n$. The problem size is set as for `eye`.

minij	Returns the symmetric positive definite matrix defined as $A_{i,j} = \min(i, j)$. The problem size is set as for `eye`.

ones	Creates a matrix with equal elements. The default value is 1, and cab be changed using `Set("a",a)`.

parter	Creates a matrix $A_{i,j} = 1/(i-j+0.5)$. This matrix is a Cauchy and a Toepliz matrix. Most of the singular values of A are very close to π. The problem size is set as for `eye`.

pei	Creates the matrix

$$A_{i,j} = \begin{cases} \alpha + 1 & \text{if } i \neq j \\ 1 & \text{if } i = j. \end{cases}.$$

The value of α can be set as `Set("a",value)`, and it defaults to 1. This matrix is singular for $\alpha = 0$ or $-n$.

recirc_2d	Returns a matrix corresponding to the finite-difference discretization of the problem

$$-\mu\Delta u + (v_x, v_y) \cdot \nabla u = f$$

on the unit square, with homogeneous Dirichlet boundary conditions. A standard 5-pt stencil is used to discretize the diffusive term, and a simple upwind stencil is used for the convective term. Here,

$$v_x = (y - 1/2)V, \qquad v_y = (1/2 - x)V$$

The value of μ can be specified using `Set("diff", DoubleValue)`, and that of V using `Set("conv", DoubleValues)`. The default values are $\mu = 10^{-5}, V = 1$. The problem size is specified as in `laplace_3d`.

`ris`	Returns a symmetric Hankel matrix with elements $A_{i,j} = 0.5/(n - i - j + 1.5)$, where n is problem size. The eigenvalues of A cluster around $-\pi/2$ and $\pi/2$.
`tridiag`	Creates a tridiagonal matrix. The diagonal element is set using `Set("a", a)`, the subdiagonal using `Set("b",b)`, and the superdiagonal using `Set("c",c)`. The default values are `a=2`, `b=c=1`. The problem size is specified as for `eye`.
`uni_flow_2d`	Returns a matrix corresponding to the finite-difference discretization of the problem $$-\mu\Delta u + (v_x, v_y) \cdot \nabla u = f$$ on the unit square, with homogeneous Dirichlet boundary conditions. A standard 5-pt stencil is used to discretize the diffusive term, and a simple upwind stencil is used for the convective term. Here, $$v_x = cos(\alpha)V, \qquad v_y = sin(\alpha)V$$ that corresponds to an unidirectional 2D flow. The value of μ can be specified using `Set("diff", DoubleValue)`, and that of V using `Set("conv", DoubleValue)`, and that of α using `Set("alpha", DoubleValue)`. The default values are $V = 1, \mu = 10^{-5}, \alpha = 0$. The problem size is specified as in `laplace3d`.

Class Trilinos_Util::VrbMatrixGallery, derived from Trilinos_Util::CrsMatrixGallery, can be used to generate VBR matrices. The class creates an Epetra_CrsMatrix (following user's defined parameters, as previously specified), then "expands" this matrix into a VBR matrix. This VBR matrix is based on an Epetra_BlockMap, based on the Epetra_Map used to define the Epetra_CrsMatrix. The number of PDE equations per node is set with parameter `num_pde_eqns`. The Epetra_CrsMatrix is expanded into a VBR matrix by replicating the matrix `num_pde_eqns` times for each equation.

A list of methods implemented by Trilinos_Util::VrbMatrixGallery is reported in Table 10.

Trilinos_Util::CrsMatrixGallery can be used in conjuction with Trilinos_Util::CommandLineParser as in the following code:

```
int main(int argc, char *argv[])
{
 #ifdef HAVE_MPI
  MPI_Init(&argc,&argv);
  Epetra_MpiComm Comm(MPI_COMM_WORLD);
```

```
GetVrbMatrix()
Returns a pointer to the internally stored VBR matrix.
GetVrbExactSolution()
Returns a pointer to the internally stored exact solution vector (as an Epetra_Vector).
GetVrbStartingSolution()
Returns a pointer to the internally stored starting solution vector (as an Epetra_Vector).
GetVrbRhs()
Returns a pointer to the internally stored right-hand side (as an Epetra_Vector).
GetVrbLinearProblem()
Returns a pointer to the internally stored Epetra_LinearProblem.
GetBlockMap()
Returns a pointer to the internally stored Epetra_BlockMap.
```

Table 10. Methods of class Trilinos_Util::VbrMatrixGallery.

```
#else
  Epetra_SerialComm Comm;
#endif

  Epetra_Time Time(Comm);

  Trilinos_Util::CommandLineParser CLP(argc,argv);
  Trilinos_Util::CrsMatrixGallery Gallery("", Comm);

  Gallery.Set(CLP);

  // get matrix
  Epetra_CrsMatrix * Matrix = Gallery.GetMatrix();
  Epetra_Vector * LHS = Gallery.GetLHS();
  Epetra_Vector * StartingSolution = Gallery.GetStartingSolution();
  Epetra_Vector * ExactSolution = Gallery.GetExactSolution();
  Epetra_LinearProblem * Problem =  Gallery.GetLinearProblem();

  // various computatons...

  // check that computed solution (in StartingSolution)
  // is close enough the ExactSolution
```

```
  double residual, diff;

  Gallery.ComputeResidual(&residual);
  Gallery.ComputeDiffBetweenStartingAndExactSolutions(&diff);

  if( Comm.MyPID()==0 )
    cout << "||b-Ax||_2 = " << residual << endl;

  if( Comm.MyPID()==0 )
    cout << "||x_exact - x||_2 = " << diff << endl;

#ifdef HAVE_MPI
  MPI_Finalize() ;
#endif

  return 0 ;

}
```

This program can be executed with the following command line:

```
[msala:triutils]> mpirun -np 4 ex1.exe -problem_type=laplace_2d \
                  -problem_size=10000
```

Matrix gallery option names shall be specified with an additional leading dash ("-"). Options values will be specified as usual.

Remark 13. *Most of the examples reported in the following chapters use both Trilinos_Util::CommandLineParser and Trilinos_Util::CrsMatrixGallery to define the distributed matrix. The user is encouraged to test a given method using matrices with different numerical properties.*

6 Iterative Solution of Linear Systems with AztecOO

The AztecOO [18] package extends the Aztec library [37]. Aztec is the legacy iterative solver at the Sandia National Laboratories. It has been extracted from the MPSalsa reacting flow code [34, 32], and it is currently installed in dozens of Sandia's applications. AztecOO extends this package, using C++ classes to enable more sophisticated uses.

AztecOO is intended for the iterative solution of linear systems of the form

$$A\,X = B, \tag{1}$$

when $A \in \mathbb{R}^{n \times n}$ is the linear system matrix, X the solution, and B the right-hand side. Although AztecOO can live independently of Epetra, in this tutorial it is supposed that A is an Epetra_RowMatrix, and both X and B are Epetra_Vector or Epetra_MultiVector objects.

The Chapter reviews the following topics:

- Outline the basic issued of the iterative solution of linear systems (in § 6.1);

- Present the basic usage of AztecOO (in § 6.2);

- Define one-level domain decomposition preconditioners (in § 6.3);

- Use of AztecOO problems as preconditioners to other AztecOO problems (in § 6.4).

6.1 Theoretical Background

Our aim is to briefly present the vocabulary and notation required to define the main features available to users. The Section is neither exhaustive, nor complete. Readers are referred to the existing literature for a comprehensive presentation (see, for instance, [4, 2, 28]).

One can distinguish between two different aspects of the iterative solution of a linear system. The first one in the particular acceleration technique for a sequence of iterations vectors, that is a technique used to construct a new approximation for the solution, with information from previous approximations. This leads to specific iteration methods, generally of Krylov type, such as conjugate gradient or GMRES. The second aspect is the transformation of the given system to one that is solved more efficiently by a particular iteration method. This is called *preconditioning*. A good preconditioner improves the convergence of the iterative method, sufficiently to overcome the extra cost of its construction and application. Indeed, without a preconditioner the iterative method may even fail to converge in practice.

The convergence of iterative methods depends on the spectral properties of the linear system matrix. The basic idea is to replace the original system (1) by the left preconditioned system,

$$P^{-1}A\,X = P^{-1}B$$

or the right preconditioned system

$$AP^{-1} PX = B$$

using the linear transformation P^{-1}, called preconditioner, in order to improve the spectral properties of the linear system matrix. In general terms, a preconditioner is any kind of transformation applied to the original system which makes it easier to solve.

From a modern perspective, the general problem of finding an efficient preconditioner is to identify a linear operator P with the following properties:

1. $P^{-1}A$ is *near* to the identity matrix

2. The cost of applying the preconditioner is relatively low.

3. Preconditioning is scalable.

The role of the P in the iterative method is simple. At each iteration, it is necessary to solve an auxiliary linear system $Pz_m = r_m$ for z_m given r_m. It is unnecessary to explicitly invert P.

It should be stressed that computing the inverse of P is not mandatory; actually, the role of P is to "preconditioning" the residual at step m, r_m, through the solution of the additional system $Pz_m = r_m$. This system $Pz_m = r_m$ should be much easier to solve than the original system.

The choice of P varies from "black-box" algebraic techniques for general matrices to "problem dependent" preconditioners that exploit special features of a particular class of problems. Although problem dependent preconditioners may be very powerful, there is still a practical need for efficient preconditioning techniques for large classes of problems. Even the "black-box" preconditioners require parameters, and suitable parameter settings do vary. Between these two extrema, there is a class of preconditioners which are "general-purpose" for a particular – although large – class of problems. These preconditioners are sometimes called "gray-box" preconditioners; users supply a little information about the matrix and the problem to be solved.

The AztecOO preconditioners include Jacobi, Gauss-Seidel, polynomial and domain decomposition based methods [35]. Furthermore preconditioners can be given to an AztecOO Krylov accelerator, by using the Trilinos packages IFPACK and ML, covered in § 7 and 9, respectively.

6.2   Basic Usage

First we delineate the steps required to apply AztecOO to a linear system. To solve a linear system with AztecOO, one must create an `Epetra_LinearProblem` object (see § 4.4) with the command

```
Epetra_LinearProblem Problem(&A,&x,&b);
```

A is an Epetra matrix, and both x and b are Epetra vectors[12]. Second create an AztecOO object,

```
AztecOO Solver(Problem);
```

Next specify how to solve the linear system. All AztecOO options are set using two vectors, one of integers and the other of doubles, as detailed in the Aztec's User Guide [37].

To choose among the different AztecOO parameters, the user can create two vectors, usually called options and params, set them to the default values, and then override with the desired parameters. Here's how to set default values:

```
int    options[AZ_OPTIONS_SIZE];
double params[AZ_PARAMS_SIZE];
AZ_defaults(options, params);

Solver.SetAllAztecOptions( options );
Solver.SetAllAztecParams( params );
```

The latter two functions copy the values of options and params to variables internal to the AztecOO object.

Alternatively, it is possible to set specific parameters without creating options and params, using the AztecOO methods SetAztecOption() and SetAztecParams(). For instance,

```
Solver.SetAztecOption( AZ_precond, AZ_Jacobi );
```

to specify a point Jacobi preconditioner. (Please see to the Aztec documentation [18] for more details about the various Aztec settings.)

Finally solve the linear system, say with a maximum of 1550 iterations and the residual error norm threshold 10^{-9}:

```
Solver.Iterate(1550,1E-9);
```

The complete code is in didasko/examples/aztec/ex1.cpp.

Note that the matrix must be in local form (that is, the command A.FillComplete() *must* have been invoked before solving the linear system). The same AztecOO linear system solution procedure applies in serial and in parallel. However for some preconditioners, the convergence rate (and the number of iterations) depends on the number of processor.

When Iterate() returns, one can query for the number of iterations performed by the linear solver using Solver.NumIters(), while Solver.TrueResidual() gives the (unscaled) norm the residual.

---

[12]At the current stage of development, AztecOO does not take advantage of the Epetra_MultiVectors. It accepts Multi_Vectors, but it will solve the linear system corresponding to the first multivector only. The Belos package implements block Krylov subspace methods that are significantly more efficient for linear systems with multiple simultaneous right-hand sides.

6.3 Overlapping Domain Decomposition Preconditioners

A one level overlapping domain decomposition preconditioners P takes the form

$$P^{-1} = \sum_{i=1}^{M} R_i^T \tilde{A}_i^{-1} R_i, \tag{2}$$

The number of subdomains is M, R_i is a rectangular Boolean matrix that restricts a global vector to the subspace defined by the interior of the ith subdomain, and $\tilde{A}_i$ approximates

$$A_i = R_i A R_i^T. \tag{3}$$

($\tilde{A}_i$ may equal A_i). Typically, $\tilde{A}_i$ differs from A_i when incomplete factorizations are used in (2) to apply $\tilde{A}_i^{-1}$, or when a matrix different from A is used in (3).

(3), The specification starts with

```
Solver.SetAztecOption( AZ_precond, AZ_dom_decomp );
```

Next if an incomplete factorization of A_i will be used, then specify its parameters:

```
Solver.SetAztecOption( AZ_subdomain_solve, AZ_ilu );
Solver.SetAztecOption( AZ_graph_fill, 1 );
```

On the other hand, exact subdomain solves [13] are specified like this:

```
Solver.SetAztecOption( AZ_subdomain_solve, AZ_lu );
```

The default amount of overlap is 0; this is equivalent to "block" Jacobi preconditioning with block size equal to the size of the subdomain. If amount of overlap is one (k),

```
Solver.SetAztecOption( AZ_overlap, 1 );
```

then the "block" is augmented by all distance one (k), neighbors in the sparse matrix graph.

Remark 14. *Two-level domain decomposition schemes [35] are available through AztecOO in conjunction with ML. Please see § 9.5.*

Remark 15. *The IFPACK Trilinos package (see § 7) computes different incomplete factorizations.*

---

[13] AztecOO must be configured with the option `--enable-aztecoo-azlu`, and the package Y12M is required.

Consider for example a Laplace equation descretized with the 5-point stencil over a regular Cartesian grid on a square. If h is the mesh size, then the condition number of the unpreconditioned system is $\mathcal{O}(h)^{-2}$. In theory [35] if H is the size of each (square) subdomain, a one level Schwarz preconditioner with minimal overlap yields a preconditioned linear system with condition number $\mathcal{O}(hH)^{-1}$. The AztecOO preconditioner delivers the Table 11 gives the estimated condition numbers for the corresponding AztecOO preconditioner.

The corresponding source code is contained in
`didasko/examples/aztecoo/ex3.cpp`
The code uses the Trilinos_Util_CrsMatrixGallery class to create the matrix. The command to estimate the condition number with $h = 1/60$ and $H = 1/3$ is

```
mpirun -np 9 ex3.exe -problem_type=laplace_2d \
       -problem_size=900 -map_type=box
```

	$h = 1/30$	$h = 1/60$	$h = 1/120$
$H = 1/3$	40.01	83.80	183.41
$H = 1/4$	51.46	106.01	223.22
$H = 1/6$	79.19	150.40	311.26
$H = 1/8$	-	191.06	403.29

Table 11. Condition number for one-level domain decomposition preconditioner on a square.

6.4 AztecOO Problems as Preconditioners

Preconditioners fall into two categories. In the first category are preconditioners that are a function of the entires of the coefficient matrix. Examples include Jacobi, Gauss-Seidel, incomplete factorizations, and domain decomposition preconditioners. The second category contains preconditioners such as polynomial preconditioners that are defined by the action of the matrix on some set of vectors. For example only category two preconditioners apply in matrix free mode. The topic of this section is another type of category two preconditioner, preconditioning one Krylov subspace method by another Krylov subspace method,

AztecOO accepts Epetra_Operator objects as preconditioners. That is any class, derived from an Epetra_Operator implementing the method `ApplyInverse()` may also be used as a preconditioner, using `SetPreOperator()`. AztecOO itself can be used to define a preconditioner for AztecOO; the class AztecOO_Operator (which takes an AztecOO object in the construction phase) is derived from Epetra_Operator.

File `didasko/examples/aztecoo/ex2.cpp` shows how to use an AztecOO solver in the preconditioning phase. The main steps are sketched here.

First, we have to specify the linear problem to be solved (set the linear operator, the solution and the right-hand side), and create an AztecOO object:

```
Epetra_LinearProblem A_Problem(&A, &x, &b);
AztecOO A_Solver(A_Problem);
```

Now, we have to define the preconditioner. For the sake of clarity, we use the same Epetra_Matrix A in the preconditioning phase. However, the two matrices may differ.

```
Epetra_CrsMatrix P(A);
```

(This operation is in general expensive as it involves the copy constructor. It is used here for the sake of clarity.) Then, we create the linear problem that will be used as a preconditioner. This takes a few steps to explain. Note that all the P prefix identifies preconditioner' objects.

1. Create the linear system solve at each preconditioning step, and and we assign the linear operator (in this case, the matrix A itself)

   ```
   Epetra_LinearProblem P_Problem;
   P_Problem.SetOperator(&P);
   ```

2. As we wish to use AztecOO to solve the preconditioner step recursively, we have to define an AztecOO object:

   ```
   AztecOO P_Solver(P_Problem);
   ```

3. Specify a particular preconditioner:

   ```
   P_Solver.SetAztecOption(AZ_precond, AZ_Jacobi);
   P_Solver.SetAztecOption(AZ_output, AZ_none);
   P_Solver.SetAztecOption(AZ_solver, AZ_cg);
   ```

4. Create an AztecOO_Operator, to set the Aztec's preconditioner with and set the users defined preconditioners:

   ```
   AztecOO_Operator
   P_Operator(&P_Solver, 10);
   A_Solver.SetPrecOperator(&P_Operator);
   ```

   (Here 10 is the maximum number of iterations of the AztecOO solver in the preconditioning phase.)

5. Solve the linear system:

   ```
   int Niters=100;
   A_Solver.SetAztecOption(AZ_kspace, Niters);
   A_Solver.SetAztecOption(AZ_solver, AZ_gmres);
   A_Solver.Iterate(Niters, 1.0E-12);
   ```

`AZ_cg`	Conjugate Gradient method (intended for symmetric positive definite matrices).
`AZ_cg_condnum`	`AZ_cg` that also estimates the condition number of the preconditioned linear system (returned in `params[AZ_condnum]`)
`AZ_gmres`	restarted Generalized Minimal Residual method
`AZ_gmres_condnum`	`AZ_gmres` that also estimates the condition number of the preconditioned linear operator (returned in `params[AZ_condnum]`)
`AZ_cgs`	Conjugate Gradient Squared method
`AZ_tfqmr`	Transpose-Free Quasi Minimal Residual method
`AZ_bicgstab`	Bi-Conjugate Gradient with Stabilization method
`AZ_lu`	Serial sparse direct linear solver available if AztecOO is configured with `--enable-aztecoo-azlu`

Table 12. `options[AZ_solver]` Choices

6.5 Concluding Remarks on AztecOO

The following methods are often used:

- `NumIters()` returns the total number of iterations performed on *this* problem;

- `TrueResidal()` returns the true unscaled residual;

- `ScaledResidual()` returns the unscaled residual;

- `SetAztecDefaults()` can be used to restore default values in the options and params vectors.

`NumIters()` is useful in performance optimization. A less costly preconditioner is viable if `NumIters()` is "small", and a more costly preconditioner may be worthwhile if `NumIters()` is "large". Many iterative methods with right preconditioning apply the residual error norm threshold to the preconditioned residual, $P^{-1}r$, instead of $r = b - Ax$. `TrueResidal()` will always return the unscaled residual norm.

7 Incomplete Factorizations with IFPACK

IFPACK provides a suite of object-oriented algebraic preconditioners for the solution of pre-conditioned iterative solvers. IFPACK offers a variety of overlapping (one-level) Schwarz preconditioners. The package uses Epetra for basic matrix-vector calculations, and accepts user matrices via an abstract matrix interface. A concrete implementation for Epetra matrices is provided. The package separates graph construction from factorization, improving performance in a substantial manner with respect to other factorization packages.

In this Chapter we discuss the use of IFPACK objects as preconditioners for AztecOO solvers. Specifically we present:

- Parallel distributed memory issues (in Section 7.2).

- Incomplete factorizations and set the notation (in Section 7.1).

- How to compute incomplete Cholesky factorizations (in Section 7.3).

- IFPACK's RILU-type factorizations (in Section 7.4).

7.1 Theoretical Background

The aim of this section is to define concepts associated with incomplete factorization methods and establish our notation. This section is not supposed to be exhaustive, nor complete on this subject. The reader is referred to the existing literature for a comprehensive presentation.

A broad class of effective preconditioners is based on incomplete factorization of the linear system matrix. Such preconditioners are often referred to as incomplete lower/upper (ILU) preconditioners. ILU preconditioning techniques lie between direct and iterative methods and provide a balance between reliability and numerical efficiency. ILU preconditioners are constructed in the factored form $P = \tilde{L}\tilde{U}$, with $\tilde{L}$ and $\tilde{U}$ being lower and upper triangular matrices. Solving with P involves two triangular solutions.

ILU preconditioners are based on the observation that, although most matrices A admit an LU factorization $A = LU$, where L is (unit) lower triangular and U is upper triangular, the factors L and U often contain too many nonzero terms, making the cost of factorization too expensive in time or memory use, or both. One type of ILU preconditioner is ILU(0), which is defined as proceeding through the standard LU decomposition computations, but keeping only those terms in $\tilde{L}$ that correspond to nonzero terms in the lower triangle of A and similarly keeping only those terms in $\tilde{U}$ that correspond to nonzero terms in the upper triangle of A. Although effective, in some cases the accuracy of the ILU(0) may be insufficient to yield an adequate rate of convergence. More accurate factorizations will differ from ILU(0) by allowing some *fill-in*. The resulting class of methods is called ILU(k), where k is the level-of-fill. A level-of-fill is attributed to each element that is processed by Gaussian elimination,

and dropping will be based on the level-of-fill. The level-of-fill should be indicative of the size of the element: the higher the level-of-fill, the smaller the elements.

Other strategies consider dropping by value – for example, dropping entries smaller than a prescribed threshold. Alternative dropping techniques can be based on the numerical size of the element to be discarded. Numerical dropping strategies generally yield more accurate factorizations with the same amount of fill-in as level-of-fill methods. The general strategy is to compute an entire row of the $\tilde{L}$ and $\tilde{U}$ matrices, and then keep only a certain number of the largest entries. In this way, the amount of fill-in is controlled; however, the structure of the resulting matrices is undefined. These factorizations are usually referred to as ILUT, and a variant which performs pivoting is called ILUTP.

When solving a single linear system, ILUT methods can be more effective than ILU(k). However, in many situations a sequence of linear systems must be solved where the pattern of the matrix A in each system is identical but the values of changed. In these situations, ILU(k) is typically much more effective because the pattern of ILU(k) will also be the same for each linear system and the overhead of computing the pattern is amortized.

7.2   Parallel Incomplete Factorizations

Parallel direct sparse solvers that compute the complete factorization $A = LU$ are effective on parallel computers. However, the effective scalability of these solvers is typically limited to a speedup of order ten, regardless of the number of processors used. Also, it is typically the factorization (constructing L and U) that exhibits the best parallel speedup. The forward and back triangular solves typically exhibit very poor parallel speedup.

The situation for ILU preconditioners is even worse. Complete factorizations can scale well because of very important graph properties that can be determined at low cost. ILU factorizations do not have the same properties, so predicting fill-in across the parallel machine is not practically possible. Also, because ILU preconditioners require repeated forward and back solves, they are more affected by the poor scalability of these operations.

Because ILU preconditioners do not scale well on parallel computers, a common practice is to perform *local* ILU factorizations. In this situation, each processor computes a factorization of a subset of matrix rows and columns independently from all other processors. This additional layer of approximation leads to a block Jacobi type of preconditioner across processors, where each block is solved using an ILU preconditioner. The difficulty with this type of preconditioner is that it tends to become less robust and require more iterations as the number of processors used increases. This effect can be offset to some extent by allowing *overlap*. Overlap refers to having processors redundantly own certain rows of the matrix for the ILU factorization. Level-1 overlap is defined so that a processor will include rows that are part of its original set. In addition, if row i is part of its original set and row i of A has a nonzero entry in column j, then row j will also be included in the factorization on that processor. Other levels of overlap are computed recursively. IFPACK supports an arbitrary level of overlap. However, level-1 is often most effective. Seldom more than 3 levels are needed.

7.3 Incomplete Cholesky Factorizations

Recall that if a matrix is symmetric positive definite, it admits a Cholesky factorization of the form $A = LL^T$, where L is lower triangular. Ifpack_CrsIct is a class for constructing and using incomplete Cholesky factorizations of an Epetra_CrsMatrix. It is built in part on top of the ICT preconditioner developed by Edmond Chow at Lawrence Livermore National Laboratory [26]. Specific factorizations depend on several parameters:

- Maximum number of entries per row/column. The factorization will contain at most this number of nonzero elements in each row/column;

- Diagonal perturbation. By default, the factorization will be computed on the input matrix. However, it is possible to modify the diagonal entries of the matrix to be factorized, via functions `SetAbsoluteThreshold()` and `SetRelativeThreshold()`. Refer to the IFPACK's documentation for more details.

It is easy to have IFPACK compute the incomplete factorization. First, define an Ifpack_CrsIct object,

```
Ifpack_CrsIct * ICT = NULL;
ICT = Ifpack_CrsIct(A,DropTol,LevelFill);
```

where `A` is an Epetra_CrsMatrix (already FillComplete'd), and `DropTop` and `LevelFill` are the drop tolerance and the level-of-fill, respectively. Then, we can set the values and compute the factors,

```
ICT->InitValues(A);
ICT->Factor();
```

IFPACK can compute the estimation of the condition number

$$cond(L_i U_i) \approx \|(LU)^{-1}e\|_\infty,$$

where $e = (1, 1, \ldots, 1)^T$. (More details can be found in the IFPACK documentation.) This estimation can be computed as follows:

```
double Condest;
ICT->Condest(false,Condest);
```

Please refer to file `didasko/examples/ifpack/ex1.cpp` for a complete example of incomplete Cholesky factorization.

7.4 RILUK Factorizations

IFPACK implements various incomplete factorization for non-symmetric matrices. In this Section, we will consider the Epetra_CrsRiluk class, that can be used to produce RILU factorization of a Epetra_CrsMatrix. The class required an Ifpack_OverlapGraph in the construction phase. This means that the factorization is split into two parts:

1. Definition of the level filled graph;

2. Computation of the factors.

This approach can significantly improve the performances of code, when an ILU preconditioner has to be computed for several matrices, with different entries but with the same sparsity pattern. An Ifpack_IlukGraph object of an Epetra matrix `A` can be constructed as follows:

```
Ifpack_IlukGraph Graph =
  Ifpack_IlukGraph(A.Graph(),LevelFill,LevelOverlap);
```

Here, `LevelOverlap` is the required overlap among the subdomains.
A call to `ConstructFilledGraph()` completes the process.

Remark 16. *An Ifpack_IlukGraph object has two Epetra_CrsGraph objects, containing the L_i and U_i graphs. Thus, it is possible to manually insert and delete graph entries in L_i and U_i via the Epetra_CrsGraphInsertIndices and RemoveIndices functions. However, in this case FillComplete must be called before the graph is used for subsequent operations.*

At this point, we can create an Ifpack_CrsRiluk object,

```
ILUK = Ifpack_CrsRiluk(Graph);
```

This phase defined the graph for the incomplete factorization, without computing the actual values of the L_i and U_i factors. Instead, this operation is accomplished with

```
int initerr = ILUK->InitValues(A);
```

The ILUK object can be used with AztecOO simply setting

```
solver.SetPrecOperator(ILUK);
```

where `solver` is an AztecOO object. The example in
`didasko/examples/ifpack/ex2.cpp` shows the use of the Ifpack_CrsRiluk class.

The application of the incomplete factors to a global vector, $z = (L_i U_i^{-1})r$, results in redundant approximation for any element of z that correspond to rows that are part of more than one local ILU factor. The OverlapMode defines how those redundant values are managed. OverlapMode is an Epetra_CombinedMode enum, that can assume the following values: `Add, Zero, Insert, Average, AbxMax`. The default is to zero out all the values of z for rows that were not part of the original matrix row distribution.

7.5 Concluding Remarks on IFPACK

More documentation on the IFPACK package can be found in [16, 14].

8   The Teuchos Utility Classes

Teuchos (pronounced "te-fos") is a collection of portable C++ tools that facilitate the development of scientific codes. Only a few of the many tools in Teuchos are mentioned in this section. For more details on all of the capabilities provided by Teuchos, please refer to the online documentation (`http://software.sandia.gov/trilinos/packages/teuchos`).

Teuchos classes have been divided between a "standard" build and an "extended" build. The "standard" build contains the general purpose templated tools like BLAS/LAPACK wrappers, parameter lists, a command-line parser, serial dense matrices, timers, flop counters, and a reference-counted pointer class. These tools are built by default when Teuchos is enabled using the configure option `--enable-teuchos`. The "extended" build contains more special purpose tools like XML parsing and MPI communicators, which can be included in the Teuchos library by using the configure option `--enable-teuchos-extended`.

In this Chapter, we will present the following "standard" build classes:

- `Teuchos::ScalarTraits` class (Section 8.1): The ScalarTraits class provides a basic interface to scalar types (float, double, complex<float>, complex<double>) that is used by the templated computational classes within Teuchos. It is the mechanism by which Teuchos' capabilities can be extended to support arbitrary precisions.

- `Teuchos::SerialDenseMatrix` class (Section 8.2): The SerialDenseMatrix is a templated version of the `Epetra_SerialDenseMatrix` class that is most often used to interface with the templated BLAS/LAPACK wrappers.

- `Teuchos::BLAS` class (Section 8.3): The BLAS class provides templated wrappers for the native BLAS library and can be extended to support arbitrary precision computations.

- `Teuchos::LAPACK` class (Section 8.4): The LAPACK class provides templated wrappers for the native LAPACK library.

- `Teuchos::ParameterList` class (Section 8.5): ParameterList is a container that can be used to group all the parameters required by a given piece of code.

- `Teuchos::RefCountPtr` class (Section 8.6): RefCountPtr is a smart reference-counted pointer class, which provides a functionality similar to the garbage collector of Java.

- `Teuchos::TimeMonitor` class (Section 8.7): TimeMonitor is a timing class that starts a timer when it is initialized and stops it when the destructor is called on the class.

- `Teuchos::CommandLineProcessor` class (Section 8.8): CommandLineProcessor is a class that helps parse command line input arguments from (`argc,argv[]`).

8.1   Teuchos::ScalarTraits

The ScalarTraits class provides a basic interface to scalar types (float, double, complex<float>, complex<double>) that is used by the templated computational classes within Teuchos. This interface includes a definition of the magnitude type and methods for obtaining random numbers, representations of zero and one, the square root, and machine-specific parameters. The Teuchos classes that utilize this scalar traits mechanism are `Teuchos::SerialDenseMatrix`, `Teuchos::BLAS`, and `Teuchos::LAPACK`.

ScalarTraits enables the extension of Teuchos' computational capabilities to any scalar type that can support its basic interface. In particular, this interface can be used for arbitrary precision scalar types. An interface to the arbitrary precision library ARPREC [3] is available if Teuchos is configured with `--enable-teuchos-arprec`. Teuchos must also be configured with the local ARPREC library paths (`--with-libs`, `--with-incdirs`, and `--with-libdirs`). To obtain more information on ARPREC or download the source code, see `http://crd.lbl.gov/~dhbailey/mpdist/`.

Remark 17. *To enable complex arithmetic (complex<float> or complex<double>) support in ScalarTraits or any dependent classes, configure Teuchos with* `--enable-teuchos-complex`.

8.2   Teuchos::SerialDenseMatrix

`Teuchos::SerialDenseMatrix` is a templated version of the SerialDenseMatrix class in `Epetra` (Chapter 3). It is most useful for interfacing with the templated BLAS and LAPACK wrappers, which will be discussed in Sections 8.3 and 8.4. However, by enabling the simple construction and manipulation of small dense matrices, the SerialDenseMatrix class has also been used as an independent tool in many Trilinos packages.

`Teuchos::SerialDenseMatrix` provides a serial interface to a small dense matrix of templated scalar type. This means a SerialDenseMatrix object can be created for any scalar type supported by Teuchos::ScalarTraits (Section 8.1). Boundschecking can be enabled for this class by configuring Teuchos with `--enable-teuchos-abc`. An exception will be thrown every time a matrix bound is violated by any method. This incurs a lot of overhead for this class, so boundschecking is only recommended as a debugging tool.

To use the Teuchos::SerialDenseMatrix class, include the header:

```
#include "Teuchos_SerialDenseMatrix.hpp"
```

Creating a double-precision matrix can be done in several ways:

```
// Create an empty matrix with no dimension
Teuchos::SerialDenseMatrix<int,double> Empty_Matrix;
// Create an empty 3x4 matrix
Teuchos::SerialDenseMatrix<int,double> My_Matrix( 3, 4 );
// Basic copy of My_Matrix
```

```
Teuchos::SerialDenseMatrix<int,double> My_Copy1( My_Matrix ),
// (Deep) Copy of principle 3x3 sub-matrix of My_Matrix
                  My_Copy2( Teuchos::Copy, My_Matrix, 3, 3 ),
// (Shallow) Copy of 2x3 sub-matrix of My_Matrix
                  My_Copy3( Teuchos::View, My_Matrix, 2, 3, 1, 1 );
```

The matrix dimensions and strided storage information can be obtained:

```
int rows = My_Copy3.numRows();  // number of rows
int cols = My_Copy3.numCols();  // number of columns
int stride = My_Copy3.stride(); // storage stride
```

Matrices can change dimension:

```
Empty_Matrix.shape( 3, 3 );       // size non-dimensional matrices
My_Matrix.reshape( 3, 3 );        // resize matrices and save values
```

Filling matrices with numbers can be done in several ways:

```
My_Matrix.random();               // random numbers
My_Copy1.putScalar( 1.0 );        // every entry is 1.0
My_Copy2(1,1) = 10.0;             // individual element access
Empty_Matrix = My_Matrix;         // copy My_Matrix to Empty_Matrix
```

Basic matrix arithmetic can be performed:

```
Teuchos::SerialDenseMatrix<int,double> My_Prod( 3, 2 );
// Matrix multiplication ( My_Prod = 1.0*My_Matrix*My_Copy^T )
My_Prod.multiply( Teuchos::NO_TRANS, Teuchos::TRANS,
                  1.0, My_Matrix, My_Copy3, 0.0 );
My_Copy2 += My_Matrix;            // Matrix addition
My_Copy2.scale( 0.5 );            // Matrix scaling
```

The pointer to the array of matrix values can be obtained:

```
double* My_Array = My_Matrix.values();   // pointer to matrix values
double* My_Column = My_Matrix[2];        // pointer to third column values
```

The norm of a matrix can be computed:

```
double norm_one = My_Matrix.normOne();        // one norm
double norm_inf = My_Matrix.normInf();        // infinity norm
double norm_fro = My_Matrix.normFrobenius();  // frobenius norm
```

Matrices can be compared:

```
// Check if the matrices are equal in dimension and values
if (Empty_Matrix == My_Matrix) {
  cout<< "The matrices are the same!" <<endl;
}
// Check if the matrices are different in dimension or values
if (My_Copy2 != My_Matrix) {
  cout<< "The matrices are different!" <<endl;
}
```

A matrix can be sent to the output stream:

```
cout<< My_Matrix << endl;
```

This section presents examples of all the methods in the `Teuchos::SerialDenseMatrix` class and can be found in `didasko/examples/teuchos/ex1.cpp`. There is also a specialization of this class for serial dense vectors that includes additional creation, accessor, arithmetic, and norm methods (`Teuchos::SerialDenseVector`).

8.3   Teuchos::BLAS

The `Teuchos::BLAS` class provides templated wrappers for the native BLAS library. This class has been written to facilitate the interface between C++ codes and BLAS, which are written in Fortran. Unfortunately, the interface between C++ and Fortran function calls is not standard across all computer platforms. The `Teuchos::BLAS` class provides C++ wrappers for BLAS kernels that are specialized during the Teuchos configuration. This insulates the rest of Teuchos and its users from the details of the Fortran to C++ translation.

The `Teuchos::BLAS` class provides C++ wrappers for a substantial subset of the BLAS kernels (Figure 1). The native BLAS library implementations of those kernels will be used for the standard scalar types (float, double, complex<float>, complex<double>). However, `Teuchos::BLAS` also has a templated version of each of these kernels. Paired with `Teuchos::ScalarTraits` (Section 8.1), the `Teuchos::BLAS` class can be extended to provide arbitrary precision computations. To use the `Teuchos::BLAS` class, include the header:

```
#include "Teuchos_BLAS.hpp"
```

Creating an instance of the BLAS class for double-precision kernels looks like:

```
Teuchos::BLAS<int, double> blas;
```

This instance provides the access to all the BLAS kernels listed in Figure 1:

```
const int n = 10;
double alpha = 2.0;
double x[ n ];
```

```
for ( int i=0; i<n; i++ ) { x[i] = i; }
blas.SCAL( n, alpha, x, 1 );
int max_idx = blas.IAMAX( n, x, 1 );
cout<< "The index of the maximum magnitude entry of x[] is the "
    <<  max_idx <<"-th and x[ " << max_idx-1 << " ] = "<< x[max_idx-1]
    << endl;
```

This is a small usage example, but its purpose is to illustrate that any of the supported BLAS kernels is a method of the `Teuchos::BLAS` class. This example can be found in `didasko/examples/teuchos/ex2.cpp`.

BLAS Kernel	Description
_ROTG	Computes a Givens plane rotation
_SCAL	Scale a vector by a constant
_COPY	Copy one vector to another
_AXPY	Add one scaled vector to another
_ASUM	Sum the absolute values of the vector entries
_DOT	Compute the dot product of two vectors
_NRM2	Compute the 2-norm of a vector
_IAMAX	Determine the index of the largest magnitude entry of a vector
_GEMV	Add a scaled matrix-vector product to another scaled vector
_TRMV	Replaces a vector with its upper/lower-triangular matrix-vector product
_GER	Updates a matrix with a scaled, rank-one outer product
_GEMM	Add a scaled matrix-matrix product to another scaled matrix
_SYMM	Add a scaled symmetric matrix-matrix product to another scaled matrix
_TRMM	Add a scaled upper/lower-triangular matrix-matrix product to another scaled matrix
_TRSM	Solves an upper/lower-triangular linear system with multiple right-hand sides

Figure 1. BLAS kernels supported by Teuchos::BLAS

8.4   Teuchos::LAPACK

The `Teuchos::LAPACK` class provides templated wrappers for the native LAPACK library. This class has been written to facilitate the interface between C++ codes and BLAS, which are written in Fortran. Unfortunately, the interface between C++ and Fortran function calls is not standard across all computer platforms. The `Teuchos::LAPACK` class provides C++ wrappers for LAPACK routines that are specialized during the Teuchos configuration. This insulates the rest of Teuchos and its users from the details of the Fortran to C++ translation.

`Teuchos::LAPACK` is a serial interface only, as LAPACK functions are. Users interested in the parallel counterpart of LAPACK, ScaLAPACK, can use the Amesos package; see Chapter 10.

The `Teuchos::LAPACK` class provides C++ wrappers for a substantial subset of the LAPACK routines (Figure 2). The native LAPACK library implementations of those kernels will be used for the standard scalar types (float, double, complex<float>, complex<double>). Unlike `Teuchos::BLAS`, the `Teuchos::LAPACK` class does not have a templated version of these routines at this time, so it cannot offer arbitrary precision computations.

To use the `Teuchos::LAPACK` class, include the header:

```
#include "Teuchos_LAPACK.hpp"
```

Creating an instance of the LAPACK class for double-precision routines looks like:

```
Teuchos::LAPACK<int, double> lapack;
```

This instance provides the access to all the LAPACK routines listed in Figure 2:

```
Teuchos::SerialDenseMatrix<int,double> My_Matrix(4,4);
Teuchos::SerialDenseVector<int,double> My_Vector(4);
My_Matrix.random();
My_Vector.random();

// Perform an LU factorization of this matrix.
int ipiv[4], info;
char TRANS = 'N';
lapack.GETRF( 4, 4, My_Matrix.values(), My_Matrix.stride(), ipiv, &info );

// Solve the linear system.
lapack.GETRS( TRANS, 4, 1, My_Matrix.values(), My_Matrix.stride(),
      ipiv, My_Vector.values(), My_Vector.stride(), &info );
```

This small example illustrates how easy it is to use the `Teuchos::LAPACK` class. Furthermore, it also exhibits the compatibility of the `Teuchos::SerialDenseMatrix` and `Teuchos::SerialDenseVector` classes with the `Teuchos::LAPACK` class. This example can be found in `didasko/examples/teuchos/ex3.cpp`.

LAPACK Routine	Description
_POTRF	Computes Cholesky factorization of a real symmetric positive definite (SPD) matrix.
_POTRS	Solves a system of linear equations where the matrix has been factored by POTRF.
_POTRI	Computes the inverse of a real SPD matrix after its been factored by POTRF.
_POCON	Estimates the reciprocal of the condition number (1-norm) of a real SPD matrix after its been factored by POTRF.
_POSV	Computes the solution to a real system of linear equations where the matrix is SPD.
_POEQU	Computes row and column scaling or equilibrating a SPD matrix and reduce its condition number.
_PORFS	Improves the computed solution to a system of linear equations where the matrix is SPD.
_POSVX	Expert SPD driver: Uses POTRF/POTRS to compute the solution to a real system of linear equations where the matrix is SPD. The system can be equilibrated (POEQU) or iteratively refined (PORFS) also.
_GELS	Solves and over/underdetermined real linear system.
_GETRF	Computes an LU factorization of a general matrix using partial pivoting.
_GETRS	Solves a system of linear equations using the LU factorization computed by GETRF.
_GETRI	Computes the inverse of a matrix using the LU factorization computed by GETRF.
_GECON	Estimates the reciprocal of the condition number of a general matrix in either the 1-norm or ∞-norm using the LU factorization computed by GETRF.
_GESV	Computes the solution of a linear system of equations.
_GEEQU	Computes row and column scaling for equilibrating a linear system, reducing its condition number.
_GERFS	Improves the computes solution to a system of linear equations and provides error bounds and backward error estimates for the solution [Use after GETRF/GETRS].
_GESVX	Expert driver: Uses GETRF/GETRS to compute the solution to a real system of linear equations, returning error bounds on the solution and a condition estimate.
_GEHRD	Reduces a real general matrix to upper Hessenberg form by orthogonal similarity transformations
_HSEQR	Compute the eigenvalues of a real upper Hessenberg matrix and, optionally, the Schur decomposition.
_GEES	Computes the real Schur form, eigenvalues, and Schur vectors of a real nonsymmetric matrix.
_GEEV	Computes the eigenvalues and, optionally, the left and/or right eigenvectors of a real nonsymmetric matrix.
_ORGHR	Generates a real orthogonal matrix which is the product of the elementary reflectors computed by GEHRD.
_ORMHR	Overwrites the general real matrix with the product of itself and the elementary reflectors computed by GEHRD.
_TREVC	Computes some or all of the right and/or left eigenvectors of a real upper quasi-triangular matrix.
_TREXC	Reorders the real Schur factorization of a real matrix via orthogonal similarity transformations.
_LARND	Returns a random number from a uniform or normal distribution.
_LARNV	Returns a vector of random numbers from a chosen distribution.
_LAMCH	Determines machine parameters for floating point characteristics.
_LAPY2	Computes $x^2 + y^2$ safely, to avoid overflow.

Figure 2. LAPACK routines supported by Teuchos::LAPACK

8.5 Teuchos::ParameterList

The `Teuchos::ParameterList` class is a C++ container of *<key, value>* pairs, where the *key* is a character string (`std::string`) and the *value* can be almost any type of C++ object. The ability to hold almost any type of C++ object as a *value* in the same list makes this class very useful for storing parameters. This parameter list can then be passed to an object, like an iterative linear solver, which will use the information to define its behavior.

The `Teuchos::ParameterList` is currently being used by several Trilinos packages. For instance, all Amesos objects (see Chapter 10) and the smoothed aggregation preconditioning object `ML_Epetra::MultiLevelPreconditioner` (see Section 9.4) are configured through a `Teuchos::ParameterList`.

Remark 18. *The parameter list stores a copy of the input object if it passed by reference. If the list is passed a pointer to an object, only the pointer is copied and not the object that it points to.*

To use the `Teuchos::ParameterList` class, include the header:

```
#include "Teuchos_ParameterList.hpp"
```

Creating an empty parameter list looks like:

```
Teuchos::ParameterList My_List;
```

Setting parameters in this list can be easily done:

```
My_List.set("Max Iters", 1550);
My_List.set("Tolerance", 1e-10);
My_List.set("Solver", "GMRES");
```

The templated "set" method should cast the input *value* to the correct data type. However, in the case where the compiler is not casting the input *value* to the expected data type, an explicit cast can be used with the "set" method:

```
My_List.set("Tolerance", (float)(1e-10));
```

A hierarchy of parameter lists can be constructed using `Teuchos::ParameterList`. This means another parameter list is a valid *value* in any parameter list. To create a sublist in a parameter list and obtain a reference to it:

```
Teuchos::ParameterList& Prec_List = My_List.sublist("Preconditioner");
```

Now this parameter list can be filled with values:

```
Prec_List.set("Type", "ILU");
Prec_List.set("Drop Tolerance", 1e-3);
```

The parameter list can be queried about the existence of a parameter, sublist, or type:

```
// Has a solver been chosen?
bool solver_defined = My_List.isParameter("Solver");
// Has a preconditioner been chosen?
bool prec_defined = My_List.isSublist("Preconditioner");
// Has a tolerance been chosen and is it a double-precision number?
bool tol_double = My_List.template isType<double>("Tolerance");
// Has a drop tolerance been chosen and is it a double-precision number?
bool dtol_double = Teuchos::isParameterType<double>(Prec_List,
                                                    "Drop Tolerance");
```

The last two methods for checking the parameter type are equivalent. There is some question as to whether the syntax of the first type-checking method (`isType`) is acceptable to older compilers. Thus, the second type-checking method (`isParameterType`) is offered as a portable alternative.

Parameters can be retrieved from the parameter list in quite a few ways:

```
// Get method that creates and sets the parameter if it doesn't exist.
int its = My_List.get("Max Iters", 1200);
// Get method that retrieves a parameter of a particular type.
float tol = My_List.template get<float>("Tolerance");
```

In the above example, the first "get" method is a safe way of obtaining a parameter when its existence is indefinite but required. The second "get" method should be used when the existense of the parameter is definite. This method will throw an exception if the parameter doesn't exist. The safest way to use the second "get" method is in a try/catch block:

```
try {
tol = My_List.template get<float>("Tolerance");
}
catch (exception& e) {
tol = 1e-6;
}
```

The second "get" method uses a syntax that may not be acceptable to older compilers. Optionally, there is another portable templated "get" function that can be used in the place of the second "get" method:

```
try {
tol = Teuchos::getParameter<float>(My_List, "Tolerance");
}
catch (exception& e) {
tol = 1e-6;
}
```

A parameter list can be sent to the output stream:

```
cout<< My_List << endl;
```

For this parameter list, the output would look like:

```
Max Iters = 1550
Preconditioner ->
  Drop Tolerance = 0.001   [unused]
  Type = ILU   [unused]
Solver = GMRES    [unused]
Tolerance = 1e-10
```

It is important to note that misspelled parameters (with additional space characters, capitalizations, etc.) may be ignored. Therefore, it is important to be aware that a given parameter has not been used. Unused parameters can be printed with method:

```
My_List.unused( cout );
```

This section presents examples of all the methods in the `Teuchos::ParameterList` class and can be found in `didasko/examples/teuchos/ex4.cpp`.

8.6 Teuchos::RefCountPtr

The `Teuchos::RefCountPtr` class is a templated class for implementing automatic garbage collection in C++ using smart, reference-counted pointers. Using this class allows one client to dynamically create an object and pass the object around to other clients without fear of memory leaks. No client is required to explicitly call delete because object will be deleted when all the clients remove their references to it. The type of garbage collection performed by `Teuchos::RefCountPtr` is similar to those found in Perl and Java.

To use the `Teuchos::RefCountPtr` class, include the header:

```
#include "Teuchos_RefCountPtr.hpp"
```

The data type used with `Teuchos::RefCountPtr` should not be a built-in data type (like `int` or `double`), this creates unnecessary overhead. Instead, it should be used to manage dynamic objects and data members that need to be shared with many clients. This means that the data type will most likely be a C++ class. Consider the class hierarchy:

```
class A {
 public:
   A() {}
   virtual ~A(){}
   virtual void f(){}
};
class B1 : virtual public A {};
class B2 : virtual public A {};
class C : public B1, public B2 {};
```

Creating a reference-counted pointer to a dynamically allocated object (of type A) can be done several ways:

```
// Create a reference-counted NULL pointer of type A.
RefCountPtr<A>          a_null_ptr;
// Create a reference-counted pointer of non-const type A.
RefCountPtr<A>           a_ptr = rcp(new A);
// Create a reference-counted pointer of const type A.
RefCountPtr<const A>     ca_ptr = rcp(new A);
// Create a const reference-counted pointer of non-const type A.
const RefCountPtr<A>     a_cptr = rcp(new A);
// Create a const reference-counted pointer of const type A.
const RefCountPtr<const A> ca_cptr = rcp(new A);
```

The `Teuchos::RefCountPtr` class can also perform implicit conversions between a derived class (`B1`) and its base class (`A`):

```
RefCountPtr<B1> b1_ptr  = rcp(new B1);
RefCountPtr<A> a_ptr1 = b1_ptr;
```

Other non-implicit type conversions like static, dynamic, or const casts can be taken care of by non-member template functions:

```
RefCountPtr<const C> c_ptr = rcp(new C);
// Implicit cast from C to B2.
RefCountPtr<const B2> b2_ptr = c_ptr;
// Safe cast, type-checked, from C to A.
RefCountPtr<const A> ca_ptr1 = rcp_dynamic_cast<const A>(c_ptr);
// Unsafe cast, non-type-checked, from C to A.
RefCountPtr<const A> ca_ptr2 = rcp_static_cast<const A>(c_ptr);
// Cast away const from B2.
RefCountPtr<B2>       nc_b2_ptr = rcp_const_cast<B2>(b2_ptr);
```

Using a reference-counted pointer is very similar to using a raw C++ pointer. Some of the operations that are common to both are:

```
RefCountPtr<A>
   a_ptr2 = rcp(new A), // Initialize reference-counted pointers.
   a_ptr3 = rcp(new A); // ""
A  *ra_ptr2 = new A,    // Initialize non-reference counted pointers.
   *ra_ptr3 = new A;    // ""
a_ptr2 = rcp(ra_ptr3);  // Assign from a raw pointer (only do this once!)
a_ptr3 = a_ptr2;        // Assign one smart pointer to another.
a_ptr2 = rcp(ra_ptr2);  // Assign from a raw pointer (only do this once!)
```

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```
a_ptr2->f();            // Access a member of A using ->
ra_ptr2->f();           // ""
*a_ptr2 = *a_ptr3;      // Dereference the objects and assign.
*ra_ptr2 = *ra_ptr3;    // ""
```

However, a reference-counted pointer cannot be used everywhere a raw C++ pointer can. For instance, these statements will not compile:

```
// Pointer arithmetic ++, --, +, - etc. not defined!
a_ptr1++;               // error
// Comparison operators ==, !=, <=, >= etc. not defined!
a_ptr1 == ra_ptr1;    // error
```

Because the two are not equivalent, the `Teuchos::RefCountPtr` class provides a way of getting the raw C++ pointer held by any `RefCountPtr<A>` object:

```
A* true_ptr = a_ptr1.get();
```

These are just some of the basic features found in the `Teuchos::RefCountPtr` class. A more extensive tutorial of this powerful tool is "in the works" and will be made available to Teuchos users as soon as it is finished. The examples presented in this section can be found in `didasko/examples/teuchos/ex5.cpp`.

8.7   Teuchos::TimeMonitor

The `Teuchos::TimeMonitor` class is a container that manages a group of timers. In this way, it can be used to keep track of timings for various phases of the code. Internally, this class holds an array of `Teuchos::Time` objects. The `Teuchos::Time` class defines a basic wall-clock timer that can `start()`, `stop()`, and return the `totalElapsedTime()`.

To use the `Teuchos::TimeMonitor` class, include the header:

```
#include "Teuchos_TimeMonitor.hpp"
```

To create a timer for the TimeMonitor to manage, call:

```
RefCountPtr<Time> FactTime = TimeMonitor::getNewTimer("Factorial Time");
```

The `getNewTimer` method creates a new reference-counted `Teuchos::Time` object and adds it to the internal array. To avoid passing this timer into each method that needs timing, consider putting it in the global scope (declare it outside of `main(argc, argv[])`). Now, when we want to time a part of the code, the appropriate timer should be used to construct a local TimeMonitor:

```
Teuchos::TimeMonitor LocalTimer(*FactTime);
```

This timer will be started during the construction of `LocalTimer` and stopped when the destructor is called on `LocalTimer`.

To obtain a summary from all the timers in the global TimeMonitor, use:

```
TimeMonitor::summarize();
```

Information from each timer can also be obtained using the methods from `Teuchos::Time`. This section presents examples of all the methods in the `Teuchos::TimeMonitor` class and can be found in `didasko/examples/teuchos/ex6.cpp`.

8.8 Teuchos::CommandLineProcessor

`Teuchos::CommandLineProcessor` is a class that helps to parse command line input arguments and set runtime options. Additionally, a CommandLineProcessor object can provide the user with a list of acceptable command line arguments, and their default values.

To use the `Teuchos::CommandLineProcessor` class, include the header:

```
#include "Teuchos_CommandLineProcessor.hpp"
```

Creating an empty command line processor looks like:

```
Teuchos::CommandLineProcessor My_CLP;
```

To set and option, it must be given a name and default value. Additionally, each option can be given a help string. Although it is not necessary, a help string aids a users comprehension of the acceptable command line arguments. Some examples of setting command line options are:

```
// Set an integer command line option.
int NumIters = 1550;
My_CLP.setOption("iterations", &NumIters, "Number of iterations");
// Set a double-precision command line option.
double Tolerance = 1e-10;
My_CLP.setOption("tolerance", &Tolerance, "Tolerance");
// Set a string command line option.
string Solver = "GMRES";
My_CLP.setOption("solver", &Solver, "Linear solver");
// Set a boolean command line option.
bool Precondition;
My_CLP.setOption("precondition","no-precondition",
                &Precondition,"Preconditioning flag");
```

There are also two methods that control the strictness of the command line processor. For a command line processor to be sensitive to any bad command line option that it does not recognize, use:

```
My_CLP.recogniseAllOptions(false);
```

Then, if the parser finds a command line option it doesn't recognize, it will throw an exception. To prevent a command line processor from throwing an exception when it encounters a unrecognized option or when the help string is printed, use:

```
My_CLP.throwExceptions(false);
```

Finally, to parse the command line, `argc` and `argv` are passed to the `parse` method:

```
My_CLP.parse( argc, argv );
```

The `--help` output for this command line processor is:

```
Usage: ./ex7.exe [options]
  options:
  --help                        Prints this help message
  --pause-for-debugging         Pauses for user input to allow
                                attaching a debugger
  --iterations        int       Number of iterations
                                (default: --iterations=1550)
  --tolerance         double    Tolerance
                                (default: --tolerance=1e-10)
  --solver            string    Linear solver
                                (default: --solver="GMRES")
  --precondition      bool      Preconditioning flag
  --no-precondition             (default: --precondition)
```

This section presents examples of all the methods in the `Teuchos::CommandLineProcessor` class and can be found in `didasko/examples/teuchos/ex7.cpp`.

9 Multilevel Preconditioners with ML

The ML package defines a class of preconditioners based on multilevel methods [6, 36]. While theoretically ML preconditioners apply to any linear system, the range of applicability of the methods is limited at this time, primarily to certain linear elliptic partial differential equations descretized with linear shape functions. The ML package provides multilevel solvers and preconditioners based on geometric and algebraic coarsening schemes. Please contact the developers for information on the status of special purpose methods, such as those for the incompressible Navier-Stokes equations and Maxwell's equations.

This Chapter will present:

- A multilevel preconditioning framework (in Section 9.1);

- How to use ml objects as AztecOO preconditioners (in Section 9.2);

- The ML_Epetra::MultiLevelOperator class (in Section 9.3);

- How to define black-box preconditioners using the ML_Epetra::MultiLevelPreconditioner Class (in Section 9.4);

- How to implement two level domain decomposition methods with aggregation based coarse matrix (in Section 9.5).

9.1 A Multilevel Preconditioning Framework

For certain combinations of iterative methods and linear systems, the error at each iteration projected onto the eigenfunctions has components that decay at a rate proportional to the corresponding eigenvalue (or frequency). Multilevel methods exploit this property [6] by projecting the linear system onto a hierarchy of increasingly coarsened "meshes" so that each error component rapidly decays on at least one coarse "mesh." The linear system on the coarsest "mesh", called the coarse grid problem, is solved exactly. The iterative method is called the smoother, as a reflection of its diminished role as a way to damp out the high frequency error. The grid transfer (or interpolation) operators are called restriction ($\mathbf{R}$) and prolongation ($\mathbf{P}$) operators.

Multilevel methods are characterized by the sequence of coarse spaces, the definition of the operator each coarse space, the specification of the smoother, and the restriction and prolongation operators. Geometric multigrid (GMG) methods are multilevel methods that require the user to specify the underlying grid, and in most cases a hierarchy of (not necessarily nested) coarsens grids. Both the automatic generation of a grid-hierarchy for GMG and the specification of the ML, designed for unstructured problems, are beyond the scope of the tutorial.

Algebraic multigrid (AMG) (see [6, Section 8]) method development has been motivated by the demand for multilevel methods that are easier to use. In AMG, both the matrix hierarchy and the prolongation operators are constructed just from the stiffness matrix. Recall that to use

Aztec00 or IFPACK, a user must supply a linear system, a select a preconditioning strategy. In AMG, the only additional information required from the user is to specify a coarsening strategy.

Readers that are unfamiliar with multigrid methods are strongly advised to review [6] before using ML.

A multilevel method for (1) depends on the number of coarsen grid levels, and operators for each level. Levels are numbered from the coarsest level, 0, to the finest. The pre- and post-smoothers are denoted $\mathbf{S}_k^{(1)}$ and $\mathbf{S}_k^{(2)}$ respectively. $\mathbf{R}_{k-1,k}$ is the restriction operator from level k to $k-1$, and $\mathbf{P}_{k,k-1}$ is a prolongator from $k-1$ to k. In AMG, the operators on the coarse meshes $\mathbf{A}_k$ are defined by

$$\mathbf{A}_{k-1} = \mathbf{R}_{k-1,k}\mathbf{A}_k\mathbf{P}_{k,k-1}.$$

The AMG coarse grid operators may be less sparse than the corresponding GMG coarse grid operators, defined by assembling the coefficient matrix on the coarse grid. These pieces combine into a multigrid solver.

<div align="center">Recursive Definition of the V-cycle Scheme <code>MGM</code>(x, b, Number of Levels)</div>

```
if k > 0
```
$$\quad x = \mathbf{S}_k^{(1)}(x, b)$$
$$\quad d = \mathbf{R}_{k-1,k}(b - \mathbf{A}_k x)$$
$$\quad v = \mathbf{0}$$
```
    MGM( v, d, k − 1)
```
$$\quad x = x + \mathbf{P}_{k,k-1}v$$
$$\quad x = \mathbf{S}_k^{(2)}(x, b)$$
```
else
```
$$\quad x = \mathbf{A}_k^{-1}b$$

Remark 19. *The tutorial only discussed AMG methods. The interested reader is referred to [38] for information on GMG methods in ML.*

9.2   ML Objects as AztecOO Preconditioners

ML may be used as a "black-box" multilevel preconditioner, using aggregation procedures to define the multilevel hierarchy. In order to use ML as a preconditioner, we need to define an AztecOO Solver, as outlined in Section 6. ML requires the user to define a structure that stores internal data. The convention in ML is to call the structure `ml_handle`. Next define the maximum number of levels, the amount of diagnostic information written to the screen, which varies from none (0) to extremely verbose (10), and creaste the ML handle structure:

```
ML *ml_handle;
```

```
int N_levels = 10;
ML_Set_PrintLevel(3);
ML_Create(&ml_handle,N_levels);
```

Next construct an ML preconditioner for an Epetra matrix. Additionally, ML requires a structure that stores information about the aggregates at each level called ML_Aggregate:

```
EpetraMatrix2MLMatrix(ml_handle, 0, &A);
ML_Aggregate *agg_object;
ML_Aggregate_Create(&agg_object);
```

The multilevel hierarchy is constructed with the instruction

```
N_levels = ML_Gen_MGHierarchy_UsingAggregation(ml_handle,
                                               0,
                                               ML_INCREASING,
                                               agg_object);
```

Here, 0 is the index of the finest level, and the index of coarser levels will be obtained by incrementing this value. Please see [29] for more information on the input parameters.

Next define the smoother, such as symmetric Gauss-Seidel, and initialize the solver:

```
ML_Gen_Smoother_SymGaussSeidel(ml_handle, ML_ALL_LEVELS,
                               ML_BOTH, 1, ML_DEFAULT);
ML_Gen_Solver (ml_handle, ML_MGV, 0, N_levels-1);
```

Finally, use this ML hierarchy to create an Epetra_Operator, set the preconditioning operator of our AztecOO solver, and then call Iterate() as usual:

```
ML_Epetra::MultiLevelOperator MLop(ml_handle,comm,map,map);
solver.SetPrecOperator(&MLop);
solver.Iterate(Niters, 1e-12);
```

The entire code is reported in
didasko/examples/ml/ex1.cpp. The output is reported below.

```
[msala:ml]> mpirun -np 2 ./ex1.exe
****************************************************************
* ML Aggregation information                                  *
================================================================
ML_Aggregate : ordering           = natural.
ML_Aggregate : min nodes/aggr      = 2
ML_Aggregate : max neigh selected  = 0
ML_Aggregate : attach scheme       = MAXLINK
ML_Aggregate : coarsen scheme      = UNCOUPLED
```

```
ML_Aggregate : strong threshold   = 0.000000e+00
ML_Aggregate : P damping factor   = 1.333333e+00
ML_Aggregate : number of PDEs     = 1
ML_Aggregate : number of null vec = 1
ML_Aggregate : smoother drop tol  = 0.000000e+00
ML_Aggregate : max coarse size    = 1
ML_Aggregate : max no. of levels  = 10
************************************************************
ML_Gen_MGHierarchy : applying coarsening
ML_Aggregate_Coarsen begins
ML_Aggregate_CoarsenUncoupled : current level = 0
ML_Aggregate_CoarsenUncoupled : current eps = 0.000000e+00
Aggregation(UVB) : Total nonzeros = 128 (Nrows=30)
Aggregation(UC) : Phase 0 - no. of bdry pts  = 0
Aggregation(UC) : Phase 1 - nodes aggregated = 28 (30)
Aggregation(UC) : Phase 1 - total aggregates = 8
Aggregation(UC_Phase2_3) : Phase 1 - nodes aggregated = 28
Aggregation(UC_Phase2_3) : Phase 1 - total aggregates = 8
Aggregation(UC_Phase2_3) : Phase 2a- additional aggregates = 0
Aggregation(UC_Phase2_3) : Phase 2 - total aggregates = 8
Aggregation(UC_Phase2_3) : Phase 2 - boundary nodes   = 0
Aggregation(UC_Phase2_3) : Phase 3 - leftovers = 0 and singletons = 0
 Aggregation time        = 1.854551e-03
Gen_Prolongator : max eigen = 1.883496e+00
ML_Gen_MGHierarchy : applying coarsening
ML_Gen_MGHierarchy : Gen_RAP
RAP time for level  0 = 5.319577e-04
ML_Gen_MGHierarchy : Gen_RAP done
ML_Gen_MGHierarchy : applying coarsening
ML_Aggregate_Coarsen begins
ML_Aggregate_CoarsenUncoupled : current level = 1
ML_Aggregate_CoarsenUncoupled : current eps = 0.000000e+00
Aggregation(UVB) : Total nonzeros = 46 (Nrows=8)
Aggregation(UC) : Phase 0 - no. of bdry pts  = 0
Aggregation(UC) : Phase 1 - nodes aggregated = 6 (8)
Aggregation(UC) : Phase 1 - total aggregates = 2
Aggregation(UC_Phase2_3) : Phase 1 - nodes aggregated = 6
Aggregation(UC_Phase2_3) : Phase 1 - total aggregates = 2
Aggregation(UC_Phase2_3) : Phase 2a- additional aggregates = 0
Aggregation(UC_Phase2_3) : Phase 2 - total aggregates = 2
Aggregation(UC_Phase2_3) : Phase 2 - boundary nodes   = 0
```

```
Aggregation(UC_Phase2_3) : Phase 3 - leftovers = 0 and singletons = 0
 Aggregation time        = 1.679042e-03
Gen_Prolongator : max eigen = 1.246751e+00
ML_Gen_MGHierarchy : applying coarsening
ML_Gen_MGHierarchy : Gen_RAP
RAP time for level  1 = 4.489557e-04
ML_Gen_MGHierarchy : Gen_RAP done
ML_Gen_MGHierarchy : applying coarsening
ML_Aggregate_Coarsen begins
Aggregation total setup time = 8.903003e-02 seconds
Smoothed Aggregation : operator complexity = 1.390625e+00.

          ****************************************************
          ***** Preconditioned CG solution
          ***** Epetra ML_Operator
          ***** No scaling
          ****************************************************

          iter:    0            residual = 1.000000e+00
          iter:    1            residual = 1.289136e-01
          iter:    2            residual = 4.710371e-03
          iter:    3            residual = 7.119470e-05
          iter:    4            residual = 1.386302e-06
          iter:    5            residual = 2.477133e-08
          iter:    6            residual = 6.141025e-10
          iter:    7            residual = 6.222216e-12
          iter:    8            residual = 1.277534e-13


          Solution time: 0.005845 (sec.)
          total iterations: 8
Residual    = 6.99704e-13
```

9.3 The ML_Epetra::MultiLevelOperator Class

As with other Trilinos packages, ML can be compiled and run independently from Epetra. It accepts input matrix in formats different from the Epetra_RowMatrix or Epetra_Operator. However, as part of the Trilinos project, ML can be used to define a preconditioner operator for `Epetra_LinearProblem` objects (see for instance [15]). This means that, in a C++ framework, ML can be defined as an `Epetra_Operator` object, applied to an `Epetra_MultiVector` object, and used as a preconditioner for AztecOO. This can be

done in two ways:

- By defining an `ML_Epetra::MultiLevelOperator` object, derived from the Epetra_Operator class. The constructor of this object requires already filled ML_Struct and ML_Aggregate structures. ML must have been configure with the option
  `--enable-epetra`.

- By defining an `ML_Epetra::MultiLevelPreconditioner` object, derived from the Epetra_RowMatrix class. Basically, the constructor of this object demands for an Epetra_RowMatrix pointer and a Teuchos parameter list, that contains all the user's defined parameters. ML must have been configure with options
  `--enable-epetra --enable-teuchos`.

The first approach, described in Section 9.3, is more general, and can be applied to geometric and algebraic multilevel preconditioner, but it requires a deeper knowledge of the ML package. This is because the user has to explicitly construct the ML hierarchy, define the aggregation strategies, the smoothers, and the coarse grid solver. The second approach, presented in Section 9.4, instead, although limited to algebraic multilevel preconditioners, allows the use of ML as a black-box preconditioner. This class automatically constructs all the components of the preconditioner, following the parameters specified in a Teuchos parameters' list.

Next we walk through how to write some code to construct an ML preconditioner for an `Epetra_RowMatrix` A. The `ML_Epetra::MultiLevelOperator` class is defined in the header file `ml_MultiLevelOperator.h`. Users must include it, and along with some subset of `ml_config.h`, `AztecOO.h`, `Epetra_Operator.h`, `Epetra_MultiVector.h` and `Epetra_LinearProblem.h`. Check the Epetra and AztecOO documentation may be helpful on this topic.

The next steps proceed exactly as in § 9.2:

```
ML *ml_handle;
int N_levels = 10;
ML_Set_PrintLevel(3);
ML_Create(&ml_handle,N_levels);
EpetraMatrix2MLMatrix(ml_handle, 0, &A);
ML_Aggregate *agg_object;
ML_Aggregate_Create(&agg_object);
N_levels = ML_Gen_MGHierarchy_UsingAggregation(ml_handle,
                                    0,
                                    ML_INCREASING,
                                    agg_object);
ML_Gen_Smoother_SymGaussSeidel(ml_handle, ML_ALL_LEVELS,
                            ML_BOTH, 1, ML_DEFAULT);
ML_Gen_Solver (ml_handle, ML_MGV, 0, N_levels-1);
```

```
ML_Epetra::MultiLevelOperator MLop(ml_handle,comm,map,map);
```

At this point, our example diverges from § 9.2. Instead of using ML to solve the linear system $Ax = b$, where x and b are `Epetra_MultiVector`, use the ML operator as the precondition for an Aztec00 linear system, and solve it:

```
Epetra_LinearProblem Problem(A,&x,&b);
AztecOO Solver(Problem);
Solver.SetPrecOperator(&MLop);
Solver.SetAztecOption( AZ_solver, AZ_gmres );
Solver.Iterate(Niters, 1e-12);
```

9.4 The ML_Epetra::MultiLevelPreconditioner Class

An alternative to the ML_Epetra::MultiLevelOperator (that is also in the namespace `ML_Epetra`) is the `MultiLevelPreconditioner` class. Replace the header file `ml_MultiLevelOperator.h` discussed in section 9.3 by `ml_MultiLevelPreconditioner.h`.

Table 13 reports the aggregation schemes currently available in ML. A graphical comparison of Uncoupled and METIS is reported in Figure 3, for a Laplacian operator over a square descretized with a a 9 point stencil.
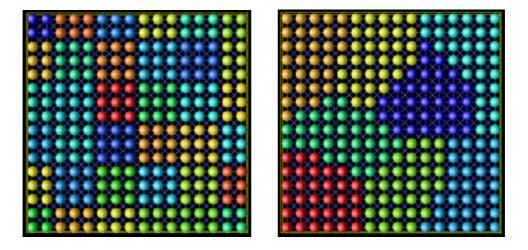

Figure 3. Aggregates for Uncoupled (left) and METIS (right) for a 16x16 Cartesian grid.

A very simple fragment of code using this class is reported below. The reader may refer to file `$ML_HOME/examples/ml_example_MultiLevelPreconditioner.cpp` for a more complex example. To run example, first configure ML `--enable-triutils`.

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Uncoupled	For a 1,2,or 3 dimensional structured Cartesian grid with a 3, 9 or 27 point stencil respectively, construct aggregates of optimal size such that each aggregate resides on one processor.
MIS	Maximal independent set based coarsening with aggregates allowed to reside on multiple processes. The scheme minimizes the number of iterations, but the cost per iteration is high.
METIS	Use a serial graph partitioner to create aggregates residing on one processor. The number of nodes in each aggregate is specified with the option `aggregation: nodes per aggregate`. ML must be configured with `--with-ml_metis`.
ParMETIS	Use a parallel graph partitioner to create aggregates that may reside on multiple processors. ML must be configured with `--with-ml_parmetis3x`. The number of aggregates is specified by option `aggregation: global number`.

Table 13. ML_Epetra::MultiLevelPreconditioner Coarsening Schemes

Jacobi	Point-Jacobi. Damping factor is specified using `smoother: damping factor`, and the number of sweeps with `smoother: sweeps`
Gauss-Seidel	Point Gauss-Seidel.
Aztec	Use AztecOO's built-in preconditioning functions as smoothers. Or, use approximate solutions with AztecOO as smoothers. The AztecOO vectors `options` and `params` can be set using `smoother: Aztec options` and `smoother: Aztec params`.

Table 14. ML_Epetra::MultiLevelPreconditioner Smoothers

Jacobi	Use Jacobi as a solver.
Gauss-Seidel	Use Gauss-Seidel as a solver.
Amesos_KLU	Use Amesos's KLU sequential solver.
Amesos_UMFPACK	Use UMFPACK.
Amesos_Superludist	Use SuperLU_DIST.
Amesos_MUMPS	Use MUMPS.

Table 15. ML_Epetra::MultiLevelPreconditioner Coarsest Grid
Exact Solvers To use Amesos, ML must be configured with
`--enable-amesos` and Amesos also be configured as needed.

```
#include "ml_include.h"
#include "ml_MultiLevelPreconditioner.h"
#include "Teuchos_ParameterList.hpp"

...

  // A is an Epetra_RowMatrix derived class object
  // solver is an AztecOO object

  Teuchos::ParameterList MList;

  // default values for smoothed aggregation
  ML_Epetra::SetDefaults("SA",MLList);
  MLList.set("max levels",6);
  MLList.set("increasing or decreasing","decreasing");
  MLList.set("aggregation: type", "MIS");
  MLList.set("coarse: type","Amesos_KLU");

  ML_Epetra::MultiLevelPreconditioner * MLPrec =
    new ML_Epetra::MultiLevelPreconditioner(A, MLList, true);

  solver.SetPrecOperator(MLPrec);
  solver.SetAztecOption(AZ_solver, AZ_gmres);
  solver.Iterate(Niters, 1e-12);

  ...
```

```
delete MLPrec;
```

The general procedure is as follows. First, the user defines a Teuchos parameters' list. Second input parameters are set via method `set(ParameterName,ParameterValue)`, where `ParameterName` is a string defining the parameter, and `ParameterValue` is the specified parameter, that can be any C++ object or pointer. This list is passed to the constructor, together with a pointer to the matrix, and a boolean flag. If this flag is set to `false`, the constructor will not compute the multilevel hierarchy until when `MLPrec->ComputePreconditioner()` is called. The hierarchy can be destroyed using `MLPrec->Destroy()`. For instance, the user may define a code like:

```
// A is still not filled with numerical values
ML_Epetra::MultiLevelPreconditioner * MLPrec =
  new ML_Epetra::MultiLevelPreconditioner(A, MLList, false);

// compute the elements of A
...
// now compute the preconditioner
MLPrec->ComputePreconditioner();

// solve the linear system, and refill A
...
MLPrec->Destroy(); // destroy previous preconditioner,
MLPrec->ComputePreconditioner(); // and build a new one
```

In this fragment of code, the user defines the ML preconditioner, but does not create the preconditioner in the construction phase. This is of particular interest, for example, when ML is used in conjunction with nonlinear solvers (like NOX [25]).

We point out that the input parameter list is *copied* in the construction phase, hence later changes to `MLList` will not affect the preconditioner. Should one need to modify parameters in the `MLPrec`'s internally stored parameter list, proceed as follows:

```
ParameterList & List = MLPrec->GetList();
```

and then directly modify `List`.

All ML options can have a common prefix, specified by the user in the construction phase. For example, suppose that we require `ML:` to be the prefix. The constructor will be

```
MLList.set("ML: aggregation: type", "METIS");
ML_Epetra::MultiLevelPreconditioner * MLPrec =
new ML_Epetra::MultiLevelPreconditioner(*A,
                                        MLList,
                                        true,
                                        Prefix);
```

where `Prefix` is a char array containing `ML: `.

Note that spaces are important: Do not include leading or trailing spaces, and separate words by just one space! Misspelled parameters will not be used, and can be detected calling method `PrintUnused()` *after* the construction of the multilevel hierarchy.

For a detailed list of all the parameters, we refer to the ML user's guide. Here, we report the most used parameters in Tables 13, 14 and 15.

9.5 Two-Level Domain Decomposition Preconditioners with ML

The idea of two level domain decomposition based on aggregation is to use a graph partitioner to partition the local or global graph into subgraphs, and then treat each subgraph as a large aggregate.

The example contained herein uses the graph decomposition library METIS to create the coarse-level matrix. If you don't have METIS, or just do not want to re-configure ML, you may run the example you will be limited to use only one aggregate per process. There are three changes to the Trilinos configuration. One flag tells the package (ML) to look for an external library, and the other two flag tells the compiler where to find the include directories and external library. Configure ML with the flags `--with-ml_metis`, and with `--with-incdirs` and `--with-ldflags` set to the locations of the METIS include files and library. Please type `configure --help` in the ML subdirectory for more information.

Two-level domain decomposition methods are effective for the iterative solution of many different kinds of linear systems. For some classes of problems, a very convenient way to define the coarse grid operator is to use an aggregation procedure. This is very close to what presented in Section 9.2. The main difference is that only two level methods are considered, and that the coarse grid remains of (relatively) small size. The idea is to define a small number of aggregates on each process, using a graph decomposition algorithm (as implemented in the library METIS, for instance)[14]. This can be done as follows.

The linear system matrix `A`, here coded as an Epetra_CrsMatrix[15], corresponds to the descretization of a 2D Laplacian on a Cartesian grid. `x` and `b` are the solution vector and the right-hand side, respectively.

The AztecOO linear problem is defined as

```
Epetra_LinearProblem problem(&A, &x, &b);
AztecOO solver(problem);
```

At this point, we can create the Teuchos parameters' list, with the following parameters:

```
ParameterList MLList;
```

---

[14] Aggregation schemes based on ParMETIS are also available. Please refer to the help of the ML `configure` for more details.

[15] Epetra_VbrMatrix and Epetra_RowMatrix can be used as well.

```
ML_Epetra::SetDefaults("DD",MLList);

MLList.set("max levels",2);
MLList.set("increasing or decreasing","increasing");

MLList.set("aggregation: type", "METIS");
MLList.set("aggregation: nodes per aggregate", 16);
MLList.set("smoother: pre or post", "both");
MLList.set("coarse: type","Amesos_KLU");
MLList.set("smoother: type", "Aztec");
```

The last option tells ML to use the Aztec preconditioning function as a smoother. Aztec requires an integer vector `options` and a double vector `params`. Those can be defined as follows:

```
int options[AZ_OPTIONS_SIZE];
double params[AZ_PARAMS_SIZE];
AZ_defaults(options,params);
options[AZ_precond] = AZ_dom_decomp;
options[AZ_subdomain_solve] = AZ_icc;
MLList.set("smoother: aztec options", options);
MLList.set("smoother: aztec params", params);
```

Note that *all* Aztec preconditioners can be used as smoothers for ML. At this point we can create the ML preconditioner as

```
ML_Epetra::MultiLevelPreconditioner * MLPrec =
  new ML_Epetra::MultiLevelPreconditioner(A, MLList, true);
```

and check that no options have been misspelled, using

```
MLPrec->PrintUnused();
```

AztecOO solver is called using

```
solver.SetPrecOperator(MLPrec);

solver.SetAztecOption(AZ_solver, AZ_cg_condnum);

int Niters = 500;
solver.SetAztecOption(AZ_kspace, 160);

solver.Iterate(Niters, 1e-12);
```

Finally, some (limited) information about the preconditioning phase are obtained using

```
cout << MLPrec->GetOutputList();
```

The entire code is reported in
`didasko/examples/ml/ex2.cpp`.

10 Interfacing Direct Solvers with Amesos

The Amesos package provides an object-oriented interface to several direct sparse solvers. Amesos will solve (using a direct factorization method) the linear systems of equations (1) where A is stored as an Epetra_RowMatrix object, and X and B are Epetra_MultiVector objects.

The Amesos package has been designed to face some of the challenges of direct solution of linear systems. In fact, many solvers have been proposed in the last years, and often each of them requires different input formats for the linear system matrix. Moreover, it is not uncommon that the interface changes between revisions. Amesos aims to solve those problems, furnishing a clean, consistent interface to many direct solvers.

Using Amesos, users can interface their codes with a (large) variety of direct linear solvers, sequential or parallel, simply by a code instruction of type

```
AmesosProblem.Solver();
```

Amesos will take care of redistributing data among the processors, if necessary.

All the Amesos classes are derived from a base class mode, `Amesos_BaseSolver`. This abstract interface provides the basic functionalities for all Amesos solvers, and allows users to choose different direct solvers very easily – by changing an input scalar parameter. See Section 10.1 for more details.

In this Chapter, we will suppose that matrix A in equation (1) is defined as an Epetra_RowMatrix, in principle with nonzero entries on all the processes defined in the Epetra_Comm communicator in use. X and B, instead, are Epetra_MultiVector, defined on the same communicator.

Amesos contains several classes:

- `Amesos_Lapack`: Interface to serial dense solver LAPACK.

- `Amesos_KLU`: Interface to Amesos's internal solver KLU. KLU is a serial, unblocked code ideal for getting started, and for very sparse matrices, such as circuit matrices.

- `Amesos_Umfpack`: Interface to Tim Davis's UMFPACK [8]. UMFPACK is a serial solver.

- `Amesos_Superludist`: Interface to Xiaoye S. Li's SuperLU solver suite, including SuperLU, SuperLU_DIST 1.0 and SuperLU_DIST 2.0 [27]. SuperLU is a serial solvers, while SuperLU_DIST is a parallel solver.

- `Amesos_Mumps`: Interface to MUMPS 4.3.1 [1][16]. MUMPS is a parallel direct solver;

---

[16]At present, MUMPS is the only Amesos class that can take advantage of the symmetry of the linear system matrix.

- `Amesos_Scalapack`: Interface to ScaLAPACK [5], the parallel version of LAPACK[17].

If the supported packages is serial, and one is solving with more than one process, matrix and right-hand side are shipped to process 0, solved, then the solution is broadcasted to the distributed solution vector X. For parallel solvers, instead, various options are supported, depending on the package at hand:

- The Amesos_Superludist interface can be used over all the processes, as well as on a subset of them. The matrix is kept in distributed form over the processes of interest;

- Amesos_Mumps can keep the matrix in a distributed form over all the processes, or the matrix can be shipped to processor 0. In both cases, all the processes in the MPI communicator will be used.

This Chapter, we will cover:

- The Amesos_BaseSolver interface to various direct solvers, presented (in Section 10.1).

10.1 Amesos_BaseSolver: A Generic Interface to Direct Solvers

All Amesos objects are constructed from the function class `Amesos`. Amesos allows a code to delay the decision about which concrete class to use to implement the Amesos_BaseSolver interface. The main goal of this class is to allow the user to select any supported (and enabled at configuration time) direct solver, simply changing an input parameter. Another remarkable advantage of Amesos_BaseSolver is that, using this class, users does not have to include the header files of the 3-part libraries in their code[18].

An example of use of this class is as follows. First, the following header files must be included:

```
#include "Amesos.h"
#include "AmesosClassType.h"
```

Then, let `A` be an Epetra_RowMatrix object (for instance, and Epetra_CrsMatrix). We need to define a linear problem,

```
Epetra_LinearProblem * Amesos_LinearProblem =
                    new Epetra_LinearProblem;
Amesos_LinearProblem->SetOperator( A ) ;
```

and to create an Amesos parameter list (which can be empty):

```
Teuchos::ParameterList ParamList ;
```

---

[17]Note that Amesos does *not* contain interfaces to LAPACK routines. Other Trilinos packages already offer those routines (Epetra and Teuchos).

[18]Using Amesos_BaseSolver, 3-part libraries header files are required in the compilation of Amesos only.

Now, let `Choice` be an string, with one of the following values:

- `AMesos_Klu`;

- `Amesos_Umfpack`;

- `Amesos_Mumps`;

- `Amesos_Superludist`;

- `Amesos_Scalapack`.

We can construct an `Amesos_BaseSolver` object as follows:

```
Amesos_BaseSolver * A_Base;
Amesos A_Factory;

A_Base = A_Factory.Create(Choice, *Amesos_LinearProblem,
                          ParamList );
assert(A_Base!=0);
```

Symbolic and numeric factorizations are computed using methods

```
A_Base->SymbolicFactorization();
A_Base->NumericFactorization();
```

The numeric factorization phase will check whether a symbolic factorization exists or not. If not, method `SymbolicFactorization()` is invoked. Solution is computed (after setting of LHS and RHS in the linear problem), using

```
A_Base->Solve();
```

The solution phase will check whether a numeric factorization exists or not. If not, method `SymbolicFactorization()` is called.

Users must provide the nonzero structure of the matrix for the symbolic phase, and the actual nonzero values for the numeric factorization. Right-hand side and solution vectors must be set before the solution phase, for instance using

```
Amesos_LinearProblem->SetLHS(x);
Amesos_LinearProblem->SetRHS(b);
```

A common ingredient to all the Amesos classes is the Teuchos parameters' list. This object, whose definition requires the input file `Teuchos_ParameterList.hpp`, is used to specify the parameters that affect the 3-part libraries. Here, we simply recall that the parameters' list can be created as

```
Teuchos::ParameterList AmesosList;
```

and parameters can be set as

```
AmesosList.set(ParameterName,ParameterValue);
```

Here, `ParameterName` is a string containing the parameter name, and `ParameterValue` is any valid C++ object that specifies the parameter value (for instance, an integer, a pointer to an array or to an object).

For a detailed list of parameters we refer to [30].

11 Eigenvalue and Eigenvector Computations with Anasazi

The Anasazi package has been designed as an extensible and interoperable framework for large-scale eigenvalue algorithms. Anasazi provides a generic interface to a collection of algorithms for solving large-scale eigenvalue problems. Anasazi requires matrices and vectors to be derived from abstract classes, and therefore it can be used with any matrix and vector classes that implement these interfaces. The algorithms currently available through Anasazi are a block Krylov Schur method and a block Davidson method.

In this Chapter we present:

- a description of the Anasazi framework (Section 11.1),

- the interface to the Epetra linear algebra package (Section 11.2), and

- an example using Anasazi for the solution of an eigenvalue problem (Section 11.3).

11.1 Anasazi Framework

The two most important classes in Anasazi are `Eigenproblem` and `Eigensolver`. The `Anasazi::Eigenproblem` class defines an interface to provide all of the information necessary to compute the solution to an eigenproblem. The `Anasazi::Eigensolver` class defines the minimum interface required of eigensolvers in Anasazi. Eigensolvers are implemented by subclassing the `Anasazi::Eigensolver` abstract class. The solvers available in the current version of Anasazi are listed in Table 16.

Solver	Description
BlockDavidson	A block Davidson solver for symmetric eigenvalue problems.
BlockKrylovSchur	A block Krylov Schur solver for symmetric or nonsymmetric problems.

Table 16. Current eigensolvers implemented in Anasazi.

Both the eigenproblem and the eigensolver in Anasazi are templated according to the scalar type, the multivector type, and the operator type. The scalar type must satisfy the interface defined by `Teuchos::ScalarTraits` (see Section 8.1). Both the operators and the multivectors are considered to be opaque objects. That is, only knowledge of the matrix and vectors via elementary operations is necessary. The required operations are specified by the interfaces `MultiVecTraits` and `OperatorTraits`. An implementation of Anasazi is accomplished via use of these interfaces.

This flexibility allows Anasazi to utilize existing linear algebra codes. For example, with regard to a multivector class, there are three choices for implementing Anasazi:

- define a specialization of an existing class which satisfies the `MultiVecTraits` class,

- create a subclass of `Anasazi::MultiVec`, for which a `MultiVecTraits` specialization is already defined by Anasazi, or

- use the Epetra class `Epetra_MultiVector`, for which a `MultiVecTraits` specialization is already defined.

Similar to the multivector class, there are three approaches for implementing Anasazi with regard to the linear operators:

- define a specialization satisfying `OperatorTraits`,

- subclass the `Anasazi::Operator` abstract class, for which a `OperatorTraits` specialization already exists, or

- use a subclass of `Epetra_Operator`, for which a `OperatorTraits` specialization is already defined.

The first and second approaches are outside the scope of this tutorial. The third of these is the simplest and allows the usage of any other Trilinos packages which recognize the Epetra multivector class. The usage of the Epetra interfaces is discussed in the next section.

Having chosen classes for the multivector and the operator, it remains to specify the operators and initial vectors to define the eigenvalue problem, to specify the options for the eigensolver, and to solve the problem. Specifying the eigenvalue problem consists mainly of setting the operators associated with the specific problem and specifying the symmetry of the problem.

For the eigensolver, we must first choose which eigensolver to use. Then any options for the eigensolver must be put into a `Teuchos::ParameterList` class (see Section 8.5) and passed to the eigensolver, along with the eigenproblem. Anasazi also makes use of the `Teuchos::RefCountPtr` smart-pointers (Section 8.6). Familiarity with these utility classes will be helpful when using Anasazi. Finally, the eigensolver may also take as an argument an instance of `Anasazi::SortManager`, to determine which eigenvalues are targeted by the solver. Usage of these classes are illustrated in Section 11.3.

11.2 Using the Epetra interface to Anasazi

In order to use the Epetra interface to Anasazi, the user must include the following files:

```
#include "AnasaziEpetraAdapter.hpp"
#include "AnasaziBasicEigenproblem.hpp"
```

In addition, the user will need one of the following, depending on whether MPI is used or not:

```
#include "Epetra_MpiComm.h"
#include "Epetra_SerialComm.h"
```

For more information about the `Epetra_Map` and `Epetra_Comm` classes, see the tutorial section covering Epetra (Sections 2.1 and 2.2).

Because Epetra makes exclusive use of double precision arithmetic, choosing `Epetra_Operator` and `Epetra_MultiVector` makes sense when the scalar type for the eigenproblem/solver is `double`. For brevity, we will declare type definitions for the multivector type and operator type:

```
typedef Epetra_MultiVector MV;
typedef Epetra_Operator OP;
```

We can now declare an eigenvalue problem and a solver, subclasses of the following abstract classes:

```
Anasazi::Eigenproblem<double,MV,OP>
Anasazi::Eigensolver<double,MV,OP>
```

An example defining an eigenvalue problem and solving the problem using an Anasazi eigensolver is given in the next section.

11.3   Defining and Solving an Eigenvalue Problem

The first step in solving an eigenvalue problem is to define the eigenvalue problem. Assume we have chosen classes to represent our scalars, multivectors and operators, as `ST`, `MV` and `OP`, respectively. Given an operator `A` and a multivector `X` to initialize the problem, both wrapped in `Teuchos::RefCountPtrs`, we might define the eigenproblem as follows:

```
Teuchos::RefCountPtr< BasicEigenproblem<ST,MV,OP> > MyProblem
  = Teuchos::rcp( new BasicEigenproblem<ST,MV,OP>(A,X) );
MyProblem->SetSymmetric( true );
MyProblem->SetNEV( 5 );
MyProblem->SetProblem();
```

The first line creates a `BasicEigenproblem` object and wraps it in a Teuchos smart-pointer. The second line specifies the symmetry of the eigenproblem, information which may be utilized by the eigensolver for extra efficiency. The third line specifies the desired number of eigenvalues and eigenvectors. Lastly, the fourth signals that we have finished setting up the eigenproblem. This step must be completed before attempting to solve the problem.

Let us, for example, attempt to solve this eigenvalue problem with a block Krylov Schur solver. First, we declare a sort manager to specify the targeted eigenvalues:

Option	Action
SM	Sort eigenvalues in increasing order of magnitude
SR	Sort eigenvalues in increasing order of real part
SI	Sort eigenvalues in increasing order of imaginary part
LM	Sort eigenvalues in decreasing order of magnitude
LR	Sort eigenvalues in decreasing order of real part
LI	Sort eigenvalues in decreasing order of imaginary part

Table 17. Options for Anasazi::BasicSort.

Option	Action
Anasazi::Error	This option is always set
Anasazi::Warning	Warnings
Anasazi::IterationDetails	Details at each iteration
Anasazi::OrthoDetails	Details about orthogonality
Anasazi::FinalSummary	A final summary
Anasazi::Debug	Debugging information

Table 18. Options for Anasazi::OutputManager.

```
std::string which("SM");
Teuchos::RefCountPtr<Anasazi::BasicSort<ST,MV,OP> > MySM =
  Teuchos::rcp( new Anasazi::BasicSort<ST,MV,OP>(which) );
```

The option chosen was "SM", signaling that we want to compute the eigenvalues with the smallest magnitude. Other options for the Anasazi::BasicSort sort manager are listed in Table 17.

Next we create an output manager, specifying the verbosity of the solver:

```
Teuchos::RefCountPtr<Anasazi::OutputManager<ST> > MyOM =
  Teuchos::rcp(  new Anasazi::OutputManager<ST>( MyPID ) );
MyOM->SetVerbosity( Anasazi::Error+Anasazi::Warning );
```

Here, we have asked for information regarding errors and warnings from the eigensolver. Other available information is listed in Table 18.

We also will specify the parameters for the eigensolver:

```
Teuchos::ParameterList MyPL;
```

Parameter	Default
Block Size	1
Max Blocks	25
Max Restarts	0
Step Size	Block_Size*Max_Blocks*(Max_Restarts+1)
Tol	1.0e-6

Table 19. Options for Anasazi::BlockKrylovSchur

Parameter	Default
Block Size	1
Max Blocks	25
Max Iters	300
Tol	1.0e-6

Table 20. Options for Anasazi::BlockDavidson

```
MyPL.set( "Block Size", 5 );
MyPL.set( "Max Blocks", 5 );
MyPL.set( "Max Restarts", 100 );
MyPL.set( "Tol", 1.0e-8 );
```

Different eigensolvers allow different parameters. Tables 19 and 20 list the settings for the block Krylov Schur and block Davidson solvers.

We now have all of the information needed to declare the eigensolver and solve the problem.

```
Anasazi::BlockKrylovSchur<ST,MV,OP>
  MySolver( MyProblem, MySM, MyOM, MyPL );
```

The eigenproblem is solved with the instruction

```
Anasazi::ReturnType sret = MySolver.solve();
```

The return value of the solver indicates whether the algorithm succeeded or not. Potential values are listed in Table 21.

A summary of results can be printed on the screen:

```
MySolver.currentStatus();
```

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Parameter	Default
Anasazi::Ok	Success.
Anasazi::Unconverged	Solver did not converge to specified accuracy.
Anasazi::Failed	An error with the input or in the solver.

Table 21. Anasazi::ReturnType values.

Eigenvectors and eigenvalues can be retrieved using

```
Teuchos::RefCountPtr<Epetra_MultiVector >  evecs
   = MyProblem->GetEvecs();
Teuchos::RefCountPtr<std::vector<double> > evals
   = MyProblem->GetEvals();
```

For a nonsymmetric problem with potentially complex eigenvalues and eigenvectors, `evecs` and `evals` contain both the real and complex parts of these, as described in the comments of `didasko/examples/anasazi/ex1.cpp`.

Example `didasko/examples/anasazi/ex1.cpp` shows how to compute the eigenvectors corresponding to the lowest eigenvalues for a 2D Laplace problem using the block Krylov Schur solver. Example `didasko/examples/anasazi/ex2.cpp` solves this example, using instead the block Davidson eigensolver. Finally, the block Krylov Schur solver is used to solve a nonsymmetric convection-diffusion problem in Example `didasko/examples/anasazi/e`

12 Solving Nonlinear Systems with NOX

NOX is a suite of solution methods for the solution of nonlinear systems of type

$$F(x) = 0, \tag{4}$$

where

$$F(x) = \begin{pmatrix} f_1(x_1, \ldots, x_n) \\ \vdots \\ f_n(x_1, \ldots, x_n) \end{pmatrix}$$

is a nonlinear vector function, and the Jacobian matrix of F, J, is defined by

$$J_{i,j} = \frac{\partial F_i}{\partial x_j}(x).$$

NOX aims to solver (4) using Newton-type methods. NOX uses an abstract vector and "group" interface. Current implementation are provided for Epetra/AztecOO objects, but also for LAPACK and PETSc. It provides various strategies for the solution of nonlinear systems, and it has been designed to be easily integrated into existing applications.

In this Chapter, we will

- Outline the basic issued of the solution of nonlinear systems (in Section 12.1);

- Introduce the NOX package (in Section 12.2);

- Describe how to introduce a NOX solver in an existing code (in Section 12.3);

- Present Jacobian-free methods (in Section 12.6).

12.1 Theoretical Background

Aim of this Section is to briefly present some aspects of the solution of nonlinear systems, to establish a notation. The Section is not supposed to be exhaustive, nor complete on this subject. The reader is referred to the existing literature for a rigorous presentation.

To solve system of nonlinear equations, NOX makes use of Newton-like methods. The Newton method defines a suite $\{x_k\}$ that, under some conditions, converges to x, solution of (4). The algorithm is as follows: given x_0, for $k = 1, \ldots$ until convergence, solve

$$J_k(x_{k-1})\,(x_k - x_{k-1}) = -F(x_{k-1}), \quad J_k(x_{k-1}) = \left[\frac{\partial F}{\partial x}(x_{k-1}) \right]. \tag{5}$$

Newton method introduces a local full linearizion of the equations. Solving a system of linear equations at each Newton step can be very expensive if the number of unknowns is large,

and may not be justified if the current iterate is far from the solution. Therefore, a departure from the Newton framework consists of considering *inexact* Newton methods, which solve system (5) only approximatively.

In fact, in practical implementation of the Newton method, one or more of the following approximations are used:

1. The Fréchet derivative J_k for the Newton step is not recomputed at every Newton step;

2. The equation for the Newton step (5) is solved only inexactly;

3. Defect-correction methods are employed, that is, J_k is numerically computed using low-order (in space) schemes, while the right-hand side is built up using high-order methods.

For a given initial guess, "close enough" to the solution of (4), the Newton method with exact linear solves converges quadratically. In practice, the radius of convergence is often extended via various methods. NOX provides, among others, line search techniques and trust region strategies.

12.2   Creating NOX Vectors and Group

NOX is not based on any particular linear algebra package. Users are required to supply methods that derive from the abstract classes `NOX::Abstract::Vector` (which provides support for basic vector operations as dot products), and `NOX::Abstract::Group` (which supports the linear algebra functionalities, evaluation of the function G and, optionally, of the Jacobian J).

In order to link their code with NOX, users have to write their own instantiation of those two abstract classes. In this tutorial, we will consider the concrete implementations provided for Epetra matrices and vectors. As this implementation is separate from the NOX algorithms, the configure option `--enable-nox-epetra` has to be specified (see Section 1.2)[19].

12.3   Introducing NOX in an Existing Code

Two basic steps are required to implement a `NOX::Epetra` interface. First, a concrete class derived from `NOX::Epetra::Interface` has to be written. This class must define the following methods:

1. A method to compute $y = F(X)$ for a given x. The syntax is

```
computeF(const Epetra_Vector & x, Epetra_Vector & y,
         FillType flag)
```

---

[19]Other two concrete implementation are provided, for LAPACK and PETSc. The user may wish to configure NOX with `--enable-nox-lapack` or `--enable-nox-petsc`. Examples can be compiled with the options `--enable-nox-lapack-examples`, `--enable-nox-petsc-examples`, and `-enable-nox-epetra-exemples`.

with `x` and `y` two Epetra_Vectors, and `flag` an enumerated type that tells why this method was called. In fact, NOX has the ability to generate Jacobians based on numerical differencing. In this case, users may want to compute an inexact (and hopefully cheaper) F, since it is only used in the Jacobian (or preconditioner).

2. A function to compute the Jacobian, whose syntax is

```
computeJacobian(const Epetra_Vector & x,
                Epetra_Operator * Jac)
```

This method is optional optional method. It should be implemented when users wish to supply their own evaluation of the Jacobian. If the user does not wish to supply their own Jacobian, they should implement this method so that it throws an error if it is called. This method should update the Jac operator so that subsequent Epetra_Operator::Apply() calls on that operator correspond to the Jacobian at the current solution vector x.

3. A method which fills a preconditioner matrix, whose syntax is

```
computePrecMatrix(const Epetra_Vector & x,
                  Epetra_RowMatrix & M)
```

It should only contain an estimate of the Jacobian. If users do not wish to supply their own Preconditioning matrix, they should implement this method such that if called, it will throw an error.

4. A method to apply the user's defined preconditioner. The syntax is

```
computePreconditioner(const Epetra_Vector & x, Epetra_Operator & M)
```

The method should compute a preconditioner based upon the solution vector x and store it in the Epetra_Operator M. Subsequent calls to the Epetra_Operator::Apply method will apply this user supplied preconditioner to epetra vectors.

Then, the user can construct a `NOX::Epetra::Group`, which contains information about the solution technique. All constructors require:

- A parameter list for printing output and for input options, defined as `NOX::Parameter::List`.

- An initial guess for the solution (stored in an Epetra_Vector object);

- an operator for the Jacobian and (optionally) and operator for the preconditioning phase. Users can write their own operators. In particular, the Jacobian can be defined by the user as an Epetra_Operator,

```
Epetra_Operator & J = UserProblem.getJacobian(),
```

created as a NOX matrix-free operator,

```
NOX::Epetra::MatrixFree & J = MatrixFree(userDefinedInterface,
                                         solutionVec),
```

or created by NOX using a finite differencing,

```
NOX::Epetra::FiniteDifference & J = FIXME...
```

At this point, users have to create an instantiation of the `NOX::Epetra::Interface` derived object,

```
UserInterface interface(...),
```

and finally construct the group,

```
NOX::Epetra::Group group(printParams, lsParams, interface).
```

12.4 A Simple Nonlinear Problem

As an example. define $F : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$F(x) = \begin{pmatrix} x_1^2 + x_2^2 - 1 \\ x_2 - x_1^2 \end{pmatrix}.$$

With this choice of F, the exact solutions of (4) are the intersections of the unity circle and the parabola $x_2 - x_1^2$. Simple algebra shows that one solution lies in the first quadrant, and has coordinates

$$\alpha = \left(\sqrt{\frac{\sqrt{5} - 1}{2}}, \frac{\sqrt{5} - 1}{2} \right),$$

the other being the reflection of α among the x_2 axis.

Code `didasko/examples/nox/ex1.cpp` applies the Newton method to this problem, with $x_0 = (0.5, 0.5)$ as a starting solution. The output is approximatively as follows:

```
[msala:nox]> mpirun -np 1 ./ex1.exe
**************************************************
-- Nonlinear Solver Step 0 --
f = 5.590e-01  step = 0.000e+00  dx = 0.000e+00
**************************************************
```

```
*****************************************************
-- Nonlinear Solver Step 1 --
f = 2.102e-01  step = 1.000e+00  dx = 3.953e-01
*****************************************************


*****************************************************
-- Nonlinear Solver Step 2 --
f = 1.009e-02  step = 1.000e+00  dx = 8.461e-02
*****************************************************


*****************************************************
-- Nonlinear Solver Step 3 --
f = 2.877e-05  step = 1.000e+00  dx = 4.510e-03 (Converged!)
*****************************************************


*****************************************************
-- Final Status Test Results --
Converged....OR Combination ->
  Converged....F-Norm = 2.034e-05 < 2.530e-04
               (Length-Scaled Two-Norm, Relative Tolerance)
  ??..........Number of Iterations = -1 < 20
*****************************************************

-- Parameter List From Solver --
Direction ->
  Method = "Newton"   [default]
  Newton ->
    Linear Solver ->
      Max Iterations = 400   [default]
      Output ->
        Achieved Tolerance = 8.6e-17   [unused]
        Number of Linear Iterations = 2   [unused]
        Total Number of Linear Iterations = 6   [unused]
      Tolerance = 1e-10   [default]
    Rescue Bad Newton Solve = true   [default]
Line Search ->
  Method = "More'-Thuente"
  More'-Thuente ->
    Curvature Condition = 1   [default]
    Default Step = 1   [default]
    Interval Width = 1e-15   [default]
```

```
    Max Iters = 20   [default]
    Maximum Step = 1e+06   [default]
    Minimum Step = 1e-12   [default]
    Optimize Slope Calculation = false   [default]
    Recovery Step = 1   [default]
    Recovery Step Type = "Constant"   [default]
    Sufficient Decrease = 0.0001   [default]
    Sufficient Decrease Condition = "Armijo-Goldstein"   [default]
  Output ->
    Total Number of Failed Line Searches = 0   [unused]
    Total Number of Line Search Calls = 3   [unused]
    Total Number of Line Search Inner Iterations = 0   [unused]
    Total Number of Non-trivial Line Searches = 0   [unused]
Nonlinear Solver = "Line Search Based"
Output ->
  2-Norm of Residual = 2.88e-05   [unused]
  Nonlinear Iterations = 3   [unused]
Printing ->
  MyPID = 0   [default]
  Output Information = 2
  Output Precision = 3   [default]
  Output Processor = 0   [default]
Computed solution :
Epetra::Vector
    MyPID           GID                 Value
        0             0                 0.786
        0             1                 0.618
Exact solution :
Epetra::Vector
    MyPID           GID                 Value
        0             0                 0.786
        0             1                 0.618
```

12.5   A 2D Nonlinear PDE

In this Section, we consider the solution of the following nonlinear PDE problem:

$$
\begin{cases}
-\Delta u + \lambda e^u &= 0 \quad \text{in } \Omega = (0,1) \times (0,1) \\
u &= 0 \quad \text{on } \partial\Omega.
\end{cases}
\tag{6}
$$

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For the sake of simplicity, we use a finite difference scheme ona Cartesian grid, with constant mesh sizes h_x and h_y. Using standard procedures, the discrete equation at node (i, j) reads

$$\frac{-u_{i-1,j} + 2u_{i,j} - u_{i+1,j}}{h_x^2} + \frac{-u_{i,j-1} + 2u_{i,j} - u_{i,j+1}}{h_y^2} - \lambda e^{u_{i,j}} = 0.$$

In example `didasko/examples/nox/ex2.cpp`, we build the Jacobian matrix as an Epetra_CrsMatrix, and we use NOX to solve problem (6) for a given value of λ. The example shows how to use NOX for more complex cases. The code defines a class, here called PDEProblem, which contains two main methods: One to compute $F(x)$ for a given x, and the other to update the entries of the Jacobian matrix. The class contains all the problem definitions (here, the number of nodes along the x-axis and the y-axis and the value of λ). In more complex cases, a similar class may have enough information to compute, for instance, the entries of J using a finite-element approximation of the PDE problem.

The interface to NOX, here called SimpleProblemInterface, accepts a PDEProblem as a constructor,

```
SimpleProblemInterface Interface(&Problem);
```

Once a NOX::Epetra:Interface object has been defined, the procedure is almost identical to that of the previous Section.

12.6 Jacobian-free Methods

In Section 12.5, the entries of the Jacobian matrix have been explicitly coded. Sometimes, it is not always possible nor convenient to compute the exact entries of J. For those cases, NOX can automatically build Jacobian matrices based on finite difference approximation, that is,

$$J_{i,j} = \frac{F_i(u + h_j e_j) - F_i(x)}{h_j},$$

where e_j is the j-unity vector. Sophisticated schemes are provided by NOX, to reduce the number of function evaluations.

12.7 Concluding Remarks on NOX

The documentation of NOX can be found in [25].

A library of continuation classes, called LOCA [31, 33], is included in the NOX distribution. LOCA is a generic continuation and bifurcation analysis package, designed for large-scalr applications.The algorithms are designed with minimal interface requirements over that needed for a Newton method to read an equilibrium solution. LOCA is built upon the NOX package. LOCA provided functionalities for single parameter continuation and multiple continuation. Also, LOCA provides a stepper class that repeatedly class the NOX nonlinear solver to compute points along a continuation curve. We will not cover LOCAL in this tutorial. The interested reader is referred to the LOCA documentation.

13 Partitioning and Load Balancing using Zoltan

In order to get good parallel performance, the data distribution (layout) is important. Poor data distribution can both cause high communication between processes and also load imbalance, that is, some processes have more work than others.

The Zoltan library [10, 9] was developed at Sandia to assist scientific computing applications with load balancing and parallel data management. Zoltan is not a Trilinos package, and must be obtained separately. The Trilinos package EpetraExt provides an interface between Epetra and Zoltan, which is described in this chapter. Note that Zoltan may be used independently of Epetra or Trilinos. Zoltan contains a collection of load-balancing algorithms with a single, data-structure neutral interface.

13.1 Background

In parallel linear algebra applications, the most critical part is to distribute the matrices. The vectors are often distributed to conform with the appropriate matrices, though not always. Matrices can be partitioned either along rows, columns, or by a 2-dimensional block decomposition. We limit our discussion to 1-dimensional data distributions (which are best supported in Epetra). In this case, partitioning dense matrices is easy. For a matrix with n rows and with p processes, simply give each process n/p rows. For sparse matrices, the situation is more complicated. To achieve load-balance, one may wish that each process obtains approximately the same number of rows, or alternatively, similar number of nonzero entries. Additionally, the communication cost when applying the matrix should be small. Specifically, for iterative solvers, the communication cost in a matrix-vector product should be minimized.

A common abstraction of this problem is *graph partitioning*. This model assumes the matrix is symmetric, so the sparsity pattern of the matrix can be represented by an undirected graph. The graph partitioning problem is to partition the vertices into p sets such that the number of edges between sets are minimized. The number of cut edges approximates the communication cost in the parallel computation. Although graph partitioning is NP-hard to solve exactly, there are several fast algorithms that work well in practice. Zoltan provides a common interface to graph partitioners (and other algorithms). At present, the most widely used software for graph partitioning, are the METIS and ParMETIS [23, 24] packages from University of Minnesota.

Recently, it has been shown [7] that hypergraph partitioning is a more accurate model for parallel matrix-vector communication cost. Work is currently underway to provide such advanced partitioning capability in future versions of Zoltan and EpetraExt. Hypergraph partitioning also applies to rectangular matrices, not just square matrices.

13.2 Installing Zoltan and Configuration

Zoltaan is not a Trilinos package and must be obtained separately. Zoltan is freely available under the LGPL license, and can be obtained from the Zoltan home page at

`http://www.cs.sandia.gov/Zoltan`.

(If you have access to the Trilinos developers cvs repository, you may also get Zoltan from the Trilinos3PL module.) Zoltan versions 1.5x are compatible with Trilinos 4.0.

We suggest that you install Zoltan together with other 3rd party software for Trilinos. It is helpful to define a shell variable `TRILINOS_3PL` for this location, see section **??** for details. After you download Zoltan, you must uncompress and untar the `Zoltan.tar.gz` file and then follow the instructions. Zoltan has not been autotool'ed so you must manually configure Zoltan to your machine. (Sample configuration files for common platforms are included in the `Utilities/Config` directory.)

Before you compile Zoltan, you need to know if you have the ParMetis library installed. ParMetis can be downloaded from
`http://www-users.cs.umn.edu/~karypis/metis/parmetis`. Version 3.1 is recommended, though other versions will work, too. It is a good idea to put the ParMetis library in the `TRILINOS_3PL` as well. You should not build more than one version of ParMetis; multiple versions may cause problems. After building ParMetis, you need to put the correct path (for your system) in the relevant Zoltan configuration file in the `Utilities/Config` directory. Then you can finally make Zoltan by typing `make zoltan`. (See the Zoltan documentation for more details.)

After you have built Zoltan, you may decide to copy the `libzoltan.a` file over to your `TRILINOS_3PL/LINUX_MPI/lib` directory (or any other name of your choice). You may also need to copy the header files in the Zoltan `include` directory to `TRILINOS_3PL/LINUX_MPI/include`. Alternatively, leave the library and the header files where they are and add these directory to the include path and library path in the Trilinos configure.

Next you need to configure Trilinos to use EpetraExt with Zoltan. You do this with a command of the type

```
./configure --prefix=${TRILINOS_HOME}/LINUX_MPI \
--enable-mpi --with-mpi-compilers \
--enable-epetraext --enable-epetraext-zoltan \
--with-trilinos3pldir=${TRILINOS_3PL} \
--with-ldflags="-L${TRILINOS_3PL}/LINUX_MPI/lib" \
--with-incdirs="-I${TRILINOS_3PL}/LINUX_MPI/include"
```

Then type `make` and `make install` as usual.

13.3 Load-balancing with Zoltan via EpetraExt

Sparse matrix computations can be load-balanced through the class
`EpetraExt::Zoltan_CrsGraph`. This is a transform that takes an Epetra_CrsGraph as input and creates a new Epetra_CrsGraph that is better load-balanced. In order to redistribute data, one needs the map of the load-balanced graph. This can be used to create an importer or exporter.

For example, say we have a Epetra_CrsMatrix that we want to load-balance. (The same procedure should work with VBR matrices, since only the graph is used for load balancing.) First create a transform:

```
EpetraExt::Zoltan_CrsGraph ZoltanTrans;
```

Next apply the transform to the graph of the matrix:

```
Epetra_CrsMatrix A;
Epetra_CrsGraph & BalGraph = ZoltanTrans(const_cast<Epetra_CrsGraph&>
    (A.Graph()));
```

The const_cast is necessary, even though normally the input graph does not change. Now we can create an exporter to export data from the old distribution to the new distribution:

```
Epetra_Export exporter(A.Graph().RowMap(), BalGraph.RowMap());
Epetra_CrsMatrix BalA(Copy, BalGraph);
BalA.Export(A, exporter, Insert);
```

For a complete example, see `didasko/examples/epetraext/ex1.cpp`. There are other (higher-level) ways to do the data rebalancing. For example, if you have a LinearProblem, use `EpetraExt::LinearProblem_GraphTrans`.

Note that currently the only load-balancing method in Zoltan supported by the EpetraExt is graph partitioning via ParMETIS. More options may be added in the future.

13.4 Load-balancing with Zoltan in other ways

Many applications use Zoltan for load-balancing directly, without going through EpetraExt. This allows a full choice of algorithms and options, while the EpetraExt interface only supports Zoltan with ParMETIS graph partitioning. Zoltan is written in ANSI C, and a lightweight C++ interface will be included in future Zoltan releases. For Trilinos users, there is already a fairly sophisticated C++ interface to Zoltan in EpetraExt. The relevant classes are `Zoltan::LoadBalance`, `Zoltan::QueryObject`, and `Zoltan::MigrationObject`. See the doxygen documentation in EpetraExt for a description on how to use these classes.

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